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Multiple Periodic Solutions for a Class of Non-Autonomous and Convex Hamiltonian Systems

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Abstract. In this paper, we study multiple periodic solutions for a class of non-autonomous and convex Hamiltonian systems and we use some properties of Ekeland index.

Keywords: Multiple Periodic Solutions, non-autonomous, Convex Hamiltonian Systems, Ekeland index.

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1. Introduction

Consider the Hamiltonian systems

$$-J\dot{u} - B(t)u = \bigtriangledown H(u,t), \tag{1}$$

where $B(t)$ is a symmetric $2N \times 2N$ matrix, continuous and T-periodic in t, $H(u,t) \in C^1(R \times R^{2N}, R)$ is a T-periodic functionin t and strictly convex, $J = \begin{pmatrix} 0 & -I_N \\ I_n & 0 \end{pmatrix}$ is the standard $2N \times 2N$ symplectic matrix, I_N is the $N \times N$ identity matrix, $\bigtriangledown H(t,u) = \frac{\partial H(,u)}{\partial u} = H'(t,u)$. Suppose that H, $\forall t \in [0,T]$ satises the conditions:

$$(A_1) : \bigtriangledown H(t,u) + B(t)u = A_0(t,u)u + o(|u|) \quad \text{as } |u| \to 0,$$
$$(A_2) : \bigtriangledown H(t,u) + B(t)u = A_\infty(t,u)u + o(|u|) \quad \text{as } |u| \to \infty.$$

where A_∞ and A_0 are $2N \times 2N$ matrix-valued continuous map of u, and are also symmet-rically positive-definite matrix for any $u \in R^{2N}$ and $|.|$ is a norm over R^{2N}.

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THEOREM 1.1 *Suppose that* $H(u,t) \in C^1(R \times R^{2N}, R)$ *satisfies:*

(1) $\frac{H(t,u)}{|u|^2} \to \infty$ *as* $|u| \to \infty$ *uniformly in* t.
(2) $\frac{H(t,u)}{|u|^2} \to 0$ *as* $|u| \to 0$ *uniformly in* t
(3) *There exist constants* $\lambda > 2$ *and* $d_1 > 0$ *such that*

$$|\bigtriangledown H(t,u)| \leqslant d_1(|u|^{\lambda-1} + 1)$$

for all $(t,u) \in [0,T] \times R^{2N}$;
(4) *There exist constants* $\beta > \lambda - 1$, $d_2 > 0$, *and* $L > 0$ *such that*

$$(\bigtriangledown H(t,u), u) - 2H(t,u) \geqslant d_2|u|^\beta, \ \forall |u| \geqslant L, \ \forall t \in [0,T]$$

If 0 *is an eigenvalue of* $-J(\frac{dH}{dt}) - B(t)$ *with periodic boundary conditions, assume also the condition:*
(5) *There exist* $\delta > 0$ *such that*
  (i) $H(t,u) \geqslant 0 \ \forall |u| \leqslant \delta, \ \forall t \in [0,T]$
  (ii) $H(t,u) \leqslant 0 \ \forall |u| \leqslant \delta, \ \forall t \in [0,T]$

Then problem (1) has at least one nontrivial T*-periodic solution, [2]. Our main result is the following theorem.*

THEOREM 1.2 *Assume* $H(u,t) \in C^1(R \times R^{2N}, R)$ *satisfies* (A_1), (A_2) *and the following three conditions:*

(A_3) $A_{01} \leqslant A_0(t,u) \leqslant A_{02}$, $i_T(A_{01}) = i_T(A_{02})$
(A_4) $A_{\infty 1} \leqslant A_\infty(t,u) \leqslant A_{\infty 2}$, $i_T(A_{\infty 1}) = i_T(A_{\infty 2})$, $\nu_T(A_{\infty 2}) = 0$
(A_5) $i_T(A_{\infty 2}) \leqslant i_T A_{02}$ *where* (A_{01}), (A_{02}), $(A_{\infty 2})$, $(A_{\infty 1})$*are symmetric positive-definite constant matrices.*

Then the system (1) has at the least

$$\frac{1}{2}[i_T(a_{02}) - i_T(A_{\infty 2})]$$

nonzero T*-periodic orbits.*

Here for any symmetrically positive definite matrix A, $i_T(A)$ *and* $\nu_T(A)$ *express its Ekeland index and nullity. We have the following formulas:*

$$i_T(A) = 2\sum_{i=1}^{N} \left\{ j \in N; \frac{2j\pi}{T} < \alpha_k \right\}, \ \nu_T(A) = 2\sum_{i=1}^{N} \left\{ j \in N; \frac{2j\pi}{T} < \alpha_k \right\}$$

where the spectrum of JA *consists of* $\pm\alpha_k$, $\alpha_k > 0$, $k = 1, 2, \ldots, n$. *Ekeland index was generally defined for any* $A \in C([O,T]; GL_s(R^{2N}))$ *satisfying* $A(t)$ *is positively definite for any* $t \in [0,T]$ *in [5].*

A brief description will be given in Section 2. where $A_{0l} = A_{02}$ and $A_{\infty 1} = A_{\infty 2}$.

2. Ekeland Index Theory

Let $L^\infty_+([0,T]; GL_s(R^N))$ be a subset of $L^\infty([0,T]; GL_s(R^N))$ such that for any clement A of that there exists an $\epsilon > 0$ satisfying $A(t) \geqslant \epsilon l_N$ for a.e $t \in [0,T]$.

Ekeland index theory is about a classication of $L_+^\infty([O,T];GL_s(R^N))$ associated to the following system:

$$J\dot{u} + A(t)u(t) = 0 \tag{2}$$

$$u(0) = u(T) \tag{3}$$

Let $\tilde{H}_T^1 = \dot{x} \in L^2([O,T];R^{2N})| \int_0^T x(t)dt = 0 \text{and} x \text{satisfies}$ (3) with inner product $(x,y) = \int_0^1 (\dot{x}(t).\dot{y}(t))dt$ and norm $||x|| = (\int_0^1 |\dot{x}(t)|^2 dt)^{1/2}$. for any $A \in L_+^\infty([0,T];GL_s(R^{2N}))$, we define $B(t) = A^{-1}(t)$ and

$$q_A(u,v) = \frac{1}{2}\left[\int_0^T (-Ju(t),\dot{v}))dt + \int_0^T (B(t)\dot{u}(t),\dot{v}(t))dt\right]; \quad \forall u,v \in \tilde{H}_T^1. \tag{4}$$

We have the following theorem:

THEOREM 2.1 *The $\tilde{H}_T^1$ can be split into three parts as follows*

$$\tilde{H}_T^1 = E^+(A) \oplus E^0(A) \oplus E^-(A)$$

such that q_T is positive definite, null and negative definite on $E^+(A)$, $E^0(A)$, and $E^-(A)$ respectively. Moreover $E^0(A)$ and $E^-(A)$ are finitely dimensional.

DEFINITION 2.2 *For any symmetric positive definite matrices of $A \in L_+^\infty([O,T];GL_s(\mathbf{R}^{2N}))$,we define*

$$\nu_T(A) = \dim E^0(A), \qquad i_T(A) = \dim E^-(A)$$

We call $\nu_T(A)$ and $i_T(A)$ the nullity and index respectively.

Ekeland index was first defined in [4] (also see [5]) for any $A \in L_+^\infty([O,T];GL_s(\mathbf{R}^{2N}))$ which is continuous for $t \in [0,T]$ and then was defined in [3] of 2006 for any $A \in L_+^\infty([O,T];GL_s(\mathbf{R}^{2N}))$. The more general Maslov type index for symplectic paths was defined in [1, 2, 7, 8]. These index theories have important application in the study of nonlinear Hamiltonian systems, [5, 8]. We see [1, 2, 6] for multiple periodic solutions of asymptotically linear Hamiltonian systems. As in [3], for any $A_1, A_2 \in Gl_s(\mathbf{R}^N)$, we write $A_l \leqslant A_2$ if $A_2 - A_l$ is positively semi-definite, and write $A_l < A_2$ if $A_2 - A_l$ is positively definite. For any $A_1, A_2 \in L^\infty((O,T);GL_s(\mathbf{R}^{2N}))$, we write $A_1 \leqslant A_2$ if $A_1(t) \leqslant A_2(t)$ for a.e $t \in (0,T)$ and write $A_1 < A_2$ if $A_1 \leqslant A_2$ and $A_1(t) < A_2(t)$ on a subset of $(0,T)$ with nonzero measure.

From [3, 4, 5] we have the following properties about Ekeland index.

PROPOSITION 2.3

1) *$i_T(A), \nu_T(A)$ are finite.*
2) *$\nu_T(A)$ is the dimension of the solution subspace of the system (2)(3).*
3) *$i_T(A) = \sum_{0<s<T} \nu_s(A)$; $i_T(A) = \sum_{0<s<1} \nu_T(sA)$.*
4) *If A_{12} then $i_T(A_1) \leqslant i_T(A_2)$ and $i_T(A_1) + \nu_T(A_1) \leqslant i_T(A_2) + \nu_T(A_2)$; if $A_1 < A_2$, then $i_T(A_1) + \nu_T(A_1) \leqslant i_T(A_2)$.*

5) There exist $\delta > 0$ such that

$$q_A(u,u) \geqslant \delta||u|| \; \forall u \in E^+(A); \quad q_A(u,u) \leqslant -\delta||u|| \quad \forall u \in E^-(A).$$

PROPOSITION 2.4 *Under assumptions (A_2) and (A_4) we have*

$$H^{*'}(t,v) = B_\infty(t,u)\nu + o(|\nu|) \quad as \quad |\nu| \to 0$$

where $B_\infty(t,u) = (A_\infty(t,u))^{-1}$, $u = H^{*'}(t,\nu)$.

Proof Set $u = H^{*'}(t,\nu)$ where $H^*(x)$ is the Legendre transform of $H(x)$. For the Legendre transform one can refer to [11, chapter 2]. By duality formula, we have $\nu = H'(t,u)$. From assumption (A_2) and (A_4) there exist positive numbers c_1 and c_2 such that

$$c_1|u| \leqslant |\nu| \leqslant c_2|u| \quad \text{as} \quad |u| \to \infty$$

We also have

$$\begin{aligned} |H^{*'}(t,\nu) - B_\infty(t,u)\nu|/|\nu| &= |u - B_\infty(t,u)H'(t,u)|/|\nu| \\ &= \{|B_\infty(t,u)(H'(t,U) - A_\infty(t,u)u)|/|u|\}|u|/|\nu| \\ &\leqslant c|(H'(t,u) - A_\infty(t,u)u|/|u| \to \text{as} \quad |\nu| \to \infty \end{aligned}$$

where $c > 0$ is a constant. ■

The following lemma is crucial for us to prove the main theorems in this paper.

LEMMA 2.5 *(Cf. Theorem 6.2 in [11].)*
Let X be a Banach space and $\varphi \in C(X,R)$ be an S^l-invariant functional satisfying (PS). Let Y and Z be closed invariant subspace of X with codim$Y$ *and* $\dim Z$ *finite and* codim$Y < \dim Z$. *Assume that the following conditions are satisfied:*

$$Fix(S^1) \subset Y, \quad Z \cap Fix(^1) = 0,$$
$$\inf_Y \varphi > -\infty$$

there exist $r > 0$ and $c < 0$ such that $\varphi(u) \leqslant c$ whenever $u \in Z$ and $||u|| = r$. If $u \in Fix(S^1)$ and $\varphi'(u) = 0$ then $\varphi(u) \leqslant 0$.

Then there exists at least $\frac{1}{2}$ ($\dim Z$ codimY) distinct S^1-orbits of critical points of φ with critical values less or equal to c.
Some concepts related are as follows.
Let G be a topological group. A representation of G over a Banach space X is a family $\{T(g)\}_{g\in G}$ of linear operators $T(g) : X \to X$ such that

$$T(0) = Id, \quad T(g_1 + g_2) = T(g_1) * T(g_2)$$
$$(g,u) \to T(g)u, \quad (g,u) \to T(g)u \quad \text{is continuous.}$$

A subset A of X is invariant (under the representation) if $T(g)A = A$ for all $g \in G$. A representation $\{T(g)\}_{g\in G}$ of G over X is isometric if $||T(g)u|| = ||u||$ for all $g \in G$ and all $u \in X$, where $||.||$ is a stand norm over X. Fix$(S^1) = \{u \in \mathbf{R}^{2N}|T(\theta)u = u \;\forall\theta \in S^1\}$, where $S^1 \simeq \mathbf{R}/T\mathbf{Z}$. A mapping M between two invariant subsets of X (under the representation of G) is equivariant if $MoT(g) = T(g)oM$ for all $g \in G$. A functional $\varphi : X \to R$ is invariant for the representation $\{T(g)\}_{g\in G}$ of topological group G if $\varphi oT(g) = \varphi$, $g \in G$. These concepts above can be found in [11, Chapters 5 and 6].

3. Proof of the Main Result

Recall that $\tilde{H}_T^1 = \{\dot{x} \in L^2([1,T];\mathbf{R}) | \int_0^T x(t)dt = 0, x(O) = x(T)\}$. Define

$$\varphi(\nu) = \int_0^T \left[\frac{1}{2}(J\dot{\nu},\nu) + H^*(\dot{\nu})\right] dt \quad \nu \in \tilde{H}_t^1 \tag{5}$$

where $H^*(t,x)$ is the Legendre transform of $H(t,x)$. It is well known that every T-periodic solution of (1) is a critical point of the functional $\varphi(\nu)$. And this functional is continuously differentiable on $\tilde{H}_t^1$ and is invariant for the representation of $S^1 \simeq \mathbf{R}/T\mathbf{Z}$ defined over $\tilde{H}_T^1$ by the translations in time $(T(\theta)\nu)(t) = \nu(t+\theta)$ if $t+\theta < T$ and $(T(\theta)\nu)(t) = \nu(t+\theta-T)$ if $t+\theta > T$ for any $\theta[0,T]$. We are in a position to apply Lemma 2.5. It is obvious that $\mathrm{Fix}(S^1) = 0$. According to the definition of $\mathrm{Fix}(S^1)$.

$$\mathrm{Fix}(S^1) = \{u | T(\theta)u = u \quad \forall\theta \in S^1 \simeq \mathbf{R}/T\mathbf{Z}\}. \tag{6}$$

By $(T(\theta)\nu)(t) = \nu(t+\theta) = \nu$, we have ν is constant function. Since $\nu \in \tilde{H}_T^1$, we have $\int_0^T \nu(t)dt = 0$ so that $\nu = 0$. It follows from (A_3) that the linear system

$$J\dot{u} + A_{01}u(t) = 0$$

has nontrivial T-periodic solution. Indeed by $\nu_T(A_{02}) = 0$, we know $\nu_T(A_{0l}) = 0$. Since H is strictly convex and $H'(0) = 0$ by (A_1), 0 is the unique equilibrium point of (1). Without loss of generality, we can assume that $H(0) = 0$. Since $H'(0) = 0$, this implies $H^*(0) = 0$.

Theorem 3.1 *very sequence (ν_m) in $\tilde{H}_T^1$ such that $\varphi'(\nu_m) \to 0$ contains a convergent subsequence [3].*

Theorem 3.2 *The functional φ is bounded from below on a closed invariant subspace Y of $\tilde{H}_T^1$ of codimension $i_T(A_\infty)$ [5].*

Theorem 3.3 *There exists an invariant subspace Z of $\tilde{H}_T^1$ with dimension $i_T(A_{01})$. and some $r > 0$ such that $\varphi(\nu) < 0$ whenever $\nu \in Z$ and $||\nu|| = r$ [5].*

Proof of Theorem 1.2

We verify assumptions of Lemma 2.5. First $\mathrm{Fix}(S^l) = 0$ and $\varphi(0) = 0$. Second φ is S^1-invariant and satisfies the (PS)-condition by Theorem 3.1. At last set $Y = E^+(A_{\infty 2})$, $Z = E^-(A_{0l})$; it follows that

$$i_T(A_{\infty 2}) = \mathrm{codim} Y < \dim Z = \dim E^-(A_{01}) = i_t(A_{01}) = i_T(A_{02})$$

Then all the assumptions of Lemma 2.5 are satisfied by Theorem 3.2 and 3.3. Thus Lemma 2.5 implies the existence of at least

$$\frac{1}{2}\left[i_T(A_{02}) - i_T(A_{\infty 2})\right]$$

distinct orbits $\{T(\theta)\nu_j : \theta \in S^l\}$ of critical points of φ outside of $\text{Fix}(S^1) = 0$. By Theorem 2.3 in [11], $u_j = H^{*'}(\dot{\nu}_j)$ is a T-periodic solution of (1) and for $j \neq k$ the u_j and u_k describe different orbits by the same argument in [11, Proof of Theorem 7.2]. This completes the proof.

4. Conclusion

In [8] has been shown the existence of multiple T-periodic solutions for the autonomous Hamiltonian systems:

$$\dot{u}(t) = J\Delta H(u(t)) \qquad u \in R^{2n}.$$

We have shown that the Hamiltonian systems:

$$-J\dot{u} - B(t)u = \Delta H(u,t),$$

where $B(t)$ is a symmetric $2N \times 2N$-matrix, continuous and T-periodic in t, $H \in C^1(R \times R^{2N}, R)$ is a T-periodic function in t and strictly convex, has at the least a nonzero T-periodic orbits. We leave the reader with special cases which without assumptions (A_3) or (A_4) or (A_5) main result is verify which has not been established yet.

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