Hierarchical Computation of Hermite Spherical Interpolant

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Abstract. In this paper, we propose to extend the hierarchical bivariate Hermite Interpolant to the spherical case. Let \(T\) be an arbitrary spherical triangle of the unit sphere \(S\) and let \(u\) be a function defined over the triangle \(T\). For \(k \in \mathbb{N}\), we consider a Hermite spherical Interpolant problem \(H_k\) defined by some data scheme \(D_k(u)\) and which admits a unique solution \(p_k\) in the space \(B_{n_k}(T)\) of homogeneous Bernstein-Bézier polynomials of degree \(n_k = 2k\) (resp. \(n_k = 2k + 1\)) defined on \(T\). We discuss the case when the data scheme \(D_r(u)\) are nested, i.e., \(D_{r-1}(u) \subseteq D_r(u)\) for all \(1 \leq r \leq k\). This, give a recursive formulae to compute the polynomial \(p_k\). Moreover, this decomposition give a new basis for the space \(B_{n_k}(T)\), which are the hierarchical structure. The method is illustrated by a simple numerical example.

Keywords: Spherical splines, Hermite interpolation, Recursive computation.

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1. Introduction

As is known, methods for building the classical univariate or bivariate Hermite spline interpolants needs the Hermite fundamental functions. In the absence of a recursive formula to calculate these basic functions, the calculation of the Hermite interpolant become difficult and complicated. To avoid this complexity Mazroui et al (see [5] and [6]) have proposed a simple, practical and useful method for calculating the Hermite interpolant recursively. More precisely, the Hermite interpolant \(p_k\) can be decomposed in the form \(p_k = p_0 + g_1 + \ldots + g_k\), where, \(p_0\) is the polynomial interpolating the set \(D_0(u)\) and \(g_r, 1 \leq r \leq k\), are particular splines.

In practice, since this decomposition make the calculation of Hermite interpolant \(p_k\) simple it can be used in the following applications, computing integrals, smoothing curves and compressing data. For more details see [5] and [6].

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Since this method is simple it is natural to extend it to several variables. One obvious way to do this is to use the tensor product. With regard to this extension, it was described in [9] (see also [7, 10]) a recursive construction for tensor product Hermite interpolants. In [8], it was proposed a method allowing to build recursively bivariate Hermite spline interpolants of class \( C^k \) on \( \mathbb{R}^2 \). Recently, we are proposed in [3] a recursive method for the construction of a Hermite spherical spline interpolant of class \( C^k \) and degree \( 4k+1 \) on \( S \). In this work, we deal with a new method allowing us to build recursively Hermite spherical interpolants on spherical triangles \( T \). In this case the degree \( n_k = 2k \) (resp. \( n_k = 2k + 1 \)) and the data scheme \( D_k(u) \) consisting of values and derivatives of \( u \) at vertices \( V_i \) and points interior \( B_j \) of triangle \( T \). But in [3], \( n_k = 4k + 1 \) and in addition to these data, we add values and derivatives of \( u \) at center points arcs \( M_{i,j} \).

Let \( T \) be a spherical triangle with vertices \( V_1, V_2 \) and \( V_3 \), and for convenience, let \( V_4 = V_1 \) and \( V_5 = V_2 \).

To define some useful derivatives associated with \( T \), let \( g_{i,j} \) be a tangent vector to \( S \) at \( v_i \) contained in the plane passing through \( v_i, v_j \), and the origin, not parallel with \( v_i, i,j = 1,2,3, \) \( i \neq j \), and for convenience, \( g_{i,0} = g_{i,3} \) and \( g_{i,4} = g_{i,1} \). In addition, let \( \mu_j, \nu_j \) be independent unit vectors lying in the tangent plane of \( S \) at \( B_j \in T \), where \( \mu_j, \nu_j \) are two non parallel directions.

Let \( u \) be a regular function defined on \( T \). For each \( k \in \mathbb{N} \), there exists a unique homogenous Bernstein-Bézier polynomial \( p_k \) in the space \( B_{n_k}(T) \) that interpolates a given data set \( D_k(u) \). In general, \( D_k(u) \) is formed by the values and the derivatives of \( u \) at the vertices \( V_i, 1 \leq i \leq 3 \), and at other points \( B_j, 1 \leq j \leq d_k \), inside \( T \) and/or on the edges of \( T \). More specifically, the set \( D_k(u) \) can be written in the form

\[
D_k(u) = \{ D^\alpha u(V_i), D^{\gamma_j} u(B_j) ; |\alpha| \leq p_k, \gamma_j \in I_k, 1 \leq i \leq 3 \},
\]

where \( I_k = \{ \gamma_j \in \mathbb{N} \times \mathbb{N}, n_{j,k} \leq |\gamma_j| \leq N_{j,k}, 1 \leq j \leq d_k, n_{j,k}, N_{j,k} \in \mathbb{N} \} \). The quantities \( D^\alpha u(V_i), \alpha = (\alpha_1, \alpha_2) \in \mathbb{N} \times \mathbb{N} \) and \( |\alpha| = \alpha_1 + \alpha_2 \), (resp. \( D^{\gamma_j} u(B_j) \), \( \gamma_j = (\gamma_{j,1}, \gamma_{j,2}) \in \mathbb{N} \times \mathbb{N} \)) denote the directional derivatives of \( u \) at \( V_i \) (resp. \( B_j \)) obtained by differentiating \( u \) \( \alpha_1 \) (resp. \( \gamma_{j,1} \)) times in the directions \( g_{i,1} \) (resp. \( \mu_j \)) and \( \alpha_2 \) (resp. \( \gamma_{j,2} \)) times in the directions \( g_{i,j-1} \) (resp. \( \nu_j \)).

Let \( H_k \) be the Hermite interpolation problem in \( B_{n_k}(T) \), corresponding to the data scheme \( D_k(u) \). Our aim is to establish a recursive formula that allows us compute step by step the polynomial \( p_k \), solution of the problem \( H_k \). This computation will be possible if some conditions are satisfied. Indeed, assume that the sets \( D_r(u), 0 \leq r \leq k \), are nested, i.e.,

\[
D_0(u) \subset D_1(u) \subset \ldots \subset D_{k-1}(u) \subset D_k(u).
\]

It is clear that (2) is equivalent to \( n_{r-1} \leq n_r, \rho_{r-1} \leq \rho_r \) and \( I_{r-1} \subset I_r \) for \( 1 \leq r \leq k \). Therefore, the polynomial \( p_k \) can be written in the form \( p_k = p_0 + q_1 + \ldots + q_k \), where each \( q_j \) is a homogenous Bernstein-Bézier polynomial of degree \( \leq n_j \) that can be determined by the data set \( D_j(u - p_{j-1}) \). The multirsolution structure of this decomposition means that \( p_0 \) may be considered as a coarse approximation of \( p_k \), and \( q_j \) are correction terms or detail polynomials. Moreover, this representation of \( p_k \) gives rise to a new basis for the space \( B_{n_k}(T) \). We show that this basis is constituted by the last Hermite basis functions of each space \( B_{n_r}(T), r = 1, \ldots, k \) and it is useful in practice.

As the bivariate case, we encounter several different Hermite interpolation problems which have unique solutions and such that their corresponding data schemes
satisfy (2). As application we deal in this works with those defined by the following two data schemes

\[ \mathcal{D}_k(u) = \{D^\alpha u(V_i), D^\gamma u(B); |\alpha| \leq k, |\gamma| \leq k - 1, i = 1,2,3\}, \quad (3) \]

\[ \mathcal{D}_k(u) = \{D^\alpha u(V_i), D^\gamma u(B); |\alpha| \leq k - 1, |\gamma| \leq k, i = 1,2,3\}, \quad (4) \]

where \( B \) is an arbitrary point inside \( T \). It is well known that there exists a unique polynomial of degree \( n_k = 2k + 1 \) (resp. \( n_k = 2k \)) that interpolates \( \mathcal{D}_k(u) \) given in (3) (resp. in (4)).

The paper is organized as follows. In Section 2, we give some preliminary results on homogeneous Bernstein-Bézier polynomials. Section 3 is devoted to the main results of this paper, namely, we first establish the hierarchical computation of Hermite polynomials \( p_{k_2}^{n_k}B_{n_k}(T) \) when corresponding data schemes \( \mathcal{D}_k(u) \) are nested. Then, for an arbitrary data schemes, we deduce a new basis for \( B_{n_k}(T) \). As an application of the above results, we describe in Section 4 the explicit decomposition of Hermite polynomial of odd or even degree that interpolate the data schemes given on (3) or (4). Finally, in Section 5 we give a numerical example.

2. Preliminary results

In this section, we present the connection between the functions defined on \( S \) and homogeneous trivariate functions, and we introduce some definitions.

A trivariate function \( F \) is said to be positively homogeneous of degree \( t \in \mathbb{R} \) provided that for every real number \( a > 0 \),

\[ F(av) = a^t F(v), \quad v \in \mathbb{R}^3 \setminus \{0\}. \]

**Lemma 2.1** (see Alfeld et al. [1]) Given a function \( f \) defined on \( S \) and let \( t \in \mathbb{R} \). Then

\[ F_t(v) = \|v\|^t f \left( \frac{v}{\|v\|} \right) \]

is the unique homogenous extension of \( f \) of degree \( t \) to all of \( \mathbb{R}^3 \setminus \{0\} \), i.e., \( F_t|_S = f \), and \( F_t \) is homogenous of degree \( t \).

Let \( g \) be a given unit vector. Then, as in [1], we define the directional derivative \( D_g \) of \( f \) at a point \( v \in S \) by

\[ D_g f(v) = D_g F(v) = g^T \nabla F(v), \]

where \( F \) is some homogenous extension of \( f \), and \( \nabla F \) is the gradient of the trivariate function \( F \).

While a polynomial of degree \( d \) has a natural homogenous extension to \( \mathbb{R}^3 \), a general function \( f \) on \( S \) has infinitely many different extensions. The value of its derivative may depend on which extension that we take (for more detail see [1]).

Let \( P_d \) be the space of trivariate polynomials of total degree at most \( d \), and let \( \mathcal{H}_d = P_d|_S \) be its restriction to the sphere \( S \). A trivariate polynomial \( p \) is called homogeneous of degree \( d \) if \( p(\lambda x, \lambda y, \lambda z) = \lambda^d p(x, y, z) \) for all \( \lambda \in \mathbb{R} \), and harmonic if \( \Delta p = 0 \), where \( \Delta \) is the Laplace operator defined by \( \Delta f = (D_x^2 + D_y^2 + D_z^2) f \).
**Definition 2.2** (see [2]) The linear space

\[ \mathcal{H}_d = \{ p|_S : p \in \mathcal{P}_d \text{ and } p \text{ is homogeneous of degree } d \text{ and harmonic} \} \]

is called the space of spherical harmonics of exact degree \( d \).

Let be given a spherical triangle \( T \). The associated spherical Bernstein basis functions of degree \( d \) are defined by

\[ B^d_{ijk}(v) = \frac{d!}{i!j!k!} b_1(v) b_2^j(v) b_3^k(v), \quad i + j + k = d, \]

where \( b_1(v), b_2(v), b_3(v) \) are spherical barycentric coordinates of \( v \) relative to \( T \). These \( \binom{d+2}{2} \) functions are linearly independent [2], and form a basis for the space denoted, in what follows, by \( B_d \). Each \( p \in B_d \) is called a spherical Bernstein-Bézier (SBB) polynomial. It is clear that \( p \) can be written in the form

\[ p = \sum_{i+j+k=d} c_{ijk} B^d_{ijk} \]

and it is uniquely determined by its B-coefficients \( c_{ijk} \).

It is well known (see [2]) that \( B^d_{ijk} \) are actually linear combinations of spherical harmonics.

**Proposition 2.3** (see [2]) For all \( d \geq 1 \), we have

\[ B_d = \begin{cases} 
\mathcal{H}_0 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_{2k} & \text{if } d = 2k, \\
\mathcal{H}_1 \oplus \mathcal{H}_3 \oplus \cdots \oplus \mathcal{H}_{2k+1} & \text{if } d = 2k + 1.
\end{cases} \]

From the above proposition, it is simple to see that

\[ B_{d-1} \not\subseteq B_d \quad \text{but} \quad B_{d-2} \subset B_d. \]

For the Hermite data scheme \( \mathcal{D}_k(u) \) given in (1), we denote by \( H_k \) the corresponding interpolation problem. Then \( H_k \) can be formulated as follows

\[ H_k \left\{ \begin{array}{l}
\text{Find } p_k \in B_{n_k}(T) \text{ such that } \\
D^\alpha p_k(V_i) = D^\alpha u(V_i), \quad |\alpha| \leq \rho_k \quad \text{and} \quad i = 1, 2, 3, \\
D^\gamma p_k(B_j) = D^\gamma u(B_j), \quad n_{j,k} \leq |\gamma| \leq N_{j,k} \quad \text{and} \quad 1 \leq j \leq d_k.
\end{array} \right. \]

**Definition 2.4** We say that \( \mathcal{D}_k(u) \) is a \( B_{n_k}(T) \)-unisolvent data scheme if the problem \( H_k \) has a unique solution \( p_k \in B_{n_k}(T) \).

In what follows, we deal with sets \( \mathcal{D}_k(u) \) that are \( B_{n_k}(T) \)-unisolvent. Then, as \( \dim B_{n_k}(T) = \binom{n_k+2}{2} \), the integers \( \rho_k, d_k, n_{j,k} \) and \( N_{j,k} \), \( 1 \leq j \leq d_k \), are given so that \( \text{card}(\mathcal{D}_k(u)) = \binom{n_k+2}{2} \).

Let \( \mathcal{B}_k = \{ \varphi^\alpha_{i,k}, \psi^\gamma_k, |\alpha| \leq \rho_k, \gamma_j \in I_k \text{ and } i = 1, 2, 3 \} \) be the Hermite basis for \( B_{n_k}(T) \) corresponding to the problem \( H_k \). More precisely, \( \varphi^\alpha_{i,k} \) and \( \psi^\gamma_k \) are determined by the following interpolation conditions

\[ \begin{align*}
D^\beta \varphi^\alpha_{i,k}(V_i) &= \delta_{i,t} \delta_{\alpha,\beta}, \quad \text{for } |\beta| \leq \rho_k \quad \text{and} \quad 1 \leq i, t \leq 3, \\
D^\gamma \varphi^\alpha_{i,k}(B_j) &= 0, \quad \text{for all } \gamma_j \in I_k, \\
D^\beta \psi^\gamma_k(V_i) &= 0, \quad \text{for all } |\beta| \leq \rho_k \quad \text{and} \quad 1 \leq t \leq 3, \\
D^\gamma \psi^\gamma_k(B_s) &= \delta_{j,s} \delta_{\gamma,\gamma_s}, \quad \text{for all } \gamma_s \in I_k,
\end{align*} \]

where \( \delta \) is the Kronecker delta.
Using the basis $B_k$, the solution $p_k$ of $H_k$ can be written in the unique form

$$p_k(\lambda) = \sum_{i=1}^{3} \sum_{|\alpha| \leq \rho_k} D^\alpha u(V_i) \varphi_i^\alpha_k(\lambda) + \sum_{\gamma_j \in I_k} D^{\gamma_j} u(B_j) \psi_k^{\gamma_j}(\lambda). \quad (6)$$

3. Recursive computation of Hermite spherical interpolants

The lack of recursive formulae for computing the basis elements of $B_k$ makes the use of (6) rather complicated. To remedy this problem, we established a decomposition of $p_k$. In other works, if we assume that $D_{k-1}(u) \subset D_k(u)$, then by using the fact that $B_{n_k}(T) \subset B_{n_k}(T)$, we deduce that $p_k = p_{k-1} + q_k$, where $p_{k-1}$ is the unique solution of the Hermite problem $H_{k-1}$ and $q_k$ is a particular polynomial in $B_{n_k}(T)$.

In order to illustrate this decomposition, we need the following lemma.

**Lemma 3.1** If $I_k \subset I_{k-1}$ and $\rho_{k-1} \leq \rho_k$, then for $|\alpha| \leq \rho_{k-1}$, $\gamma_j \in I_{k-1}$ and $i = 1, 2, 3$ we have

$$\phi_{i,k}^\alpha = \phi_{i,k-1}^\alpha - \phi_{i,k}^{\alpha_k} \text{ and } \psi_k^{\gamma_j} = \psi_{k-1}^{\gamma_j} - \psi_k^{\gamma_j},$$

where

$$\phi_{i,k}^{\alpha_k} = \sum_{l=1}^{3} \sum_{|\beta| = \rho_{k-1} + 1}^{\rho_k} D^\beta \phi_{i,k-1}^\alpha(V_l) \varphi_{l,k}^\beta + \sum_{\gamma_j \in I_k \setminus I_{k-1}} D^{\gamma_j} \phi_{i,k-1}(B_j) \psi_{k}^{\gamma_j}. \quad (7)$$

$$\psi_k^{\gamma_j} = \sum_{l=1}^{3} \sum_{|\beta| = \rho_{k-1} + 1}^{\rho_k} D^\beta \psi_{k-1}^{\gamma_j}(V_l) \varphi_{l,k}^\beta + \sum_{\gamma_j \in I_k \setminus I_{k-1}} D^{\gamma_j} \psi_{k-1}(B_j) \psi_{k}^{\gamma_j}. \quad (8)$$

**Proof** Let $I_k$ be the Hermite interpolation operator defined for a function $u$ by $I_k u = u_k \in B_{n_k}$. As $I_k$ is exact on $B_{n_k}$, i.e., $I_k p = p$ for all $p \in B_{n_k}$, we deduce that $I_k \phi_{i,k-1}^\alpha = \phi_{i,k-1}^\alpha$. In other words, we have

$$\phi_{i,k-1}^\alpha = \sum_{l=1}^{3} \sum_{|\beta| \leq \rho_k} D^\beta \phi_{i,k-1}^\alpha(V_l) \varphi_{l,k}^\beta + \sum_{\gamma_j \in I_k} D^{\gamma_j} \phi_{i,k-1}(B_j) \psi_{k}^{\gamma_j}. \quad (9)$$

On the other hand, from (5), we deduce that for all $\alpha \leq \rho_{k-1}$

$$\sum_{l=1}^{3} \sum_{|\beta| \leq \rho_{k-1}} D^\beta \phi_{i,k-1}^\alpha(V_l) \varphi_{l,k}^\beta = \phi_{i,k}^\alpha \text{ and } \sum_{\gamma_j \in I_k} D^{\gamma_j} \phi_{i,k-1}(B_j) \psi_{k}^{\gamma_j} = 0.$$

where after, we get the first equality. Using a similar technique, one can establish the other equalities. $\blacksquare$

Now, we give the main result of this paper.

**Theorem 3.2** Let $p_k$ and $p_{k-1}$ be the Hermite spherical polynomial solutions of problems $H_k$ and $H_{k-1}$ respectively. If $P_{k-1}(u) \subset D_k(u)$, then the spherical polynomial $p_k$ can be decomposed as follows

$$p_k = p_{k-1} + q_k,$$

where
\[ q_k = \sum_{i=1}^{3} \sum_{|\alpha| = \rho_{k-1}+1}^{\rho_k} D^\alpha (u - p_{k-1})(V_i) \varphi_{i,k}^\alpha + \sum_{\gamma_j \in I_k \setminus I_{k-1}} D^{\gamma_j} (u - p_{k-1})(B_j) \psi_{k}^{\gamma_j}. \]

**Proof** Recall that \( D_{k-1}(u) \subset D_k(u) \) implies that \( I_{k-1} \subset I_k \) and \( \rho_{k-1} \leq \rho_k \). Then, the expression of the Hermite polynomial \( p_k \) given in (6) becomes

\[ p_k = \sum_{i=1}^{3} \sum_{|\alpha| = \rho_{k-1}}^{\rho_k} D^\alpha u(V_i) \varphi_{i,k}^\alpha + \sum_{\gamma_j \in I_k \setminus I_{k-1}} D^{\gamma_j} u(B_j) \psi_{k}^{\gamma_j} + \]

\[ 3 \sum_{i=1}^{3} \sum_{|\alpha| = \rho_{k-1}+1}^{\rho_k} D^\alpha u(V_i) \varphi_{i,k}^\alpha + \sum_{\gamma_j \in I_k \setminus I_{k-1}} D^{\gamma_j} u(B_j) \psi_{k}^{\gamma_j}. \]

Using the expressions for \( \varphi_{i,k}^\alpha, \psi_{k}^{\gamma_j}, \varphi_{i,k}^\alpha \) and \( \psi_{k}^{\gamma_j} \), given in Lemma (3.1), we get

\[ p_k = p_{k-1} + \sum_{i=1}^{3} \sum_{|\alpha| = \rho_{k-1}+1}^{\rho_k} D^\alpha (u - p_{k-1})(V_i) \varphi_{i,k}^\alpha + \sum_{\gamma_j \in I_k \setminus I_{k-1}} D^{\gamma_j} (u - p_{k-1})(B_j) \psi_{k}^{\gamma_j}. \]

**Remark 1** From the above expression for \( q_k \), we deduce that its corresponding insolvent data set is \( D_k(u - p_{k-1}) \).

**Corollary 3.3** Assume that \( D_0(u) \subset D_1(u) \subset \ldots \subset D_k(u) \). Then we have the spherical polynomial \( p_k \) can be decomposed in the form

\[ p_k = p_0 + q_1 + \ldots + q_k, \]  

(9)

where \( q_s = \sum_{i=1}^{3} \sum_{|\alpha| = \rho_{s-1}+1}^{\rho_s} C_{i,s}^\alpha \varphi_{i,s}^\alpha + \sum_{\gamma_j \in I_k \setminus I_{s-1}} \tilde{C}_{s}^{\gamma_j} \psi_{s}^{\gamma_j}, \) \( 1 \leq s \leq k \) and \( p_0 \) is the solution of the Hermite problem \( H_0 \), and the coefficients

\[ C_{i,s}^\alpha = D^\alpha (u - p_{s-1})(V_i), \quad \tilde{C}_{s}^{\gamma_j} = D^{\gamma_j} (u - p_{s-1})(B_j) \]

can be computed recursively as follows:

for \( s = 1, |\alpha| = \rho_0 + 1, \ldots, \rho_1 \) and \( \gamma_j \in I_1 \setminus I_0, \)

\[ C_{i,1}^\alpha = D^\alpha u(V_i) - D^\alpha p_0(V_i), \quad \tilde{C}_{1}^{\gamma_j} = D^{\gamma_j} u(B_j) - D^{\gamma_j} p_0(B_j). \]
and for \( s \geq 2 \), \( |\alpha| = \rho_{s-1} + 1, \ldots, \rho_s \) and \( \gamma_j \in I_s \backslash I_{s-1} \),

\[
C_{i,s}^\alpha = \sum_{l=1}^{s-1} \left[ \sum_{l=1}^{3} \sum_{|\beta| = \rho_{l-1} + 1}^{\rho_l} C_{l,t}^\beta \varphi_{l,t}^\beta(V_i) + \sum_{\gamma_m \in I_l \backslash I_{l-1}} \tilde{\Gamma}_{l,t}^m D^\alpha \psi_l^m(V_i) \right]
\]

Then, by using the obvious equality \( C_{i,s}^\alpha = D^\alpha u(V_i) - D^\alpha p_{s-1}(V_i) \), we deduce that

\[
C_{i,s}^\alpha = D^\alpha u(V_i) - D^\alpha p_0(V_i) - \sum_{l=1}^{s-1} \left[ \sum_{l=1}^{3} \sum_{|\beta| = \rho_{l-1} + 1}^{\rho_l} C_{l,t}^\beta \varphi_{l,t}^\beta(V_i) + \sum_{\gamma_m \in I_l \backslash I_{l-1}} \tilde{\Gamma}_{l,t}^m D^\alpha \psi_l^m(V_i) \right].
\]

In the same way, we can obtain the recursive formula for \( \tilde{\Gamma}_{s}^{\alpha,j} \).

**Proof** The decomposition (9) of \( p_k \) follows from Theorem 3.2. On the other hand, it is clear that for \( s = 1, \ldots, k \), we have

\[
p_{s-1} = p_0 + \sum_{l=1}^{s-1} \left[ \sum_{l=1}^{3} \sum_{|\beta| = \rho_{l-1} + 1}^{\rho_l} C_{l,t}^\beta \varphi_{l,t}^\beta + \sum_{\gamma_m \in I_l \backslash I_{l-1}} \tilde{\Gamma}_{l,t}^m \psi_l^m \right].
\]

Now, if we put \( \rho_{-1} = -1 \) and \( I_{-1} = \emptyset \), then we have the following result.

**Theorem 3.4** The family

\[
\tilde{B}_k = \{ \varphi_{i,s}^\alpha, \psi_s^\gamma \}, \ 1 \leq i \leq 3, \ 0 \leq s \leq k, \ \rho_{s-1} + 1 \leq |\alpha| \leq \rho_s \ and \ \gamma_j \in I_s \backslash I_{s-1} \}
\]

forms a basis for the space \( B_{n_k}(T) \). Moreover, \( \tilde{B}_k, k \in \mathbb{N}, \) are hierarchical.

**Proof** Let \( p \in B_{n_k}(T) \). Since the Hermite interpolation interpolation operator \( I_k \) is exact on \( B_{n_k}(T) \), we deduce that \( p = I_k(p) = p_0 + q_1 + \cdots + q_k \), where \( p_0 \) is the unique solution of the Hermite problem \( H_0 \), and \( q_s, 1 \leq s \leq k \), are particular polynomials in \( B_{n_k}(T) \) defined by

\[
q_s = \sum_{i=1}^{3} \sum_{|\alpha| = \rho_0}^{\rho_s} \mu_{i,0}^\alpha \varphi_{i,0}^\alpha(\lambda) + \sum_{\gamma_j \in I_0}^{\sigma_0^\gamma} \psi_0^\gamma(\lambda)
\]

\[
+ \sum_{s=1}^{k-1} \left[ \sum_{i=1}^{3} \sum_{|\alpha| = \rho_{s-1} + 1}^{\rho_s} \mu_{i,s}^\alpha \varphi_{i,s}^\alpha(\lambda) + \sum_{\gamma_j \in I_s \backslash I_{s-1}}^{\sigma_s^\gamma} \psi_s^\gamma(\lambda) \right] = 0.
\]
Using the definitions of Hermite basis functions given in (5), and starting from \( s = 0 \) to \( s = k \) and from \( |\alpha| = \rho_s - 1 + 1 \) to \( |\alpha| = \rho_s \), we obtain step by step \( D^\alpha f(A_i) = \mu_{i,s}^\alpha = 0 \) and \( D^\gamma_j f(B_j) = \sigma_s^\gamma_j = 0 \). Consequently, \( \mathcal{B}_k \) is a basis for \( \mathcal{B}_k \).

On the other hand, if we put \( \mathcal{B}_0 = \mathcal{B}_0 \), it is simple to check that \( \mathcal{B}_k = \mathcal{B}_{k-1} \cup \mathcal{B}_k \), where

\[
\mathcal{B}_k = \{ \phi_{i,k}^\alpha, \psi_k^{\gamma_j}, 1 \leq i \leq 3, k \rho_{k-1} + 1 \leq |\alpha| \leq \rho_k \text{ and } \gamma_j \in I_k \setminus I_{k-1} \}.
\]

Then we have \( \mathcal{B}_{k-1} \subset \mathcal{B}_k \).

Remark 2

The comparison of the two bases \( \mathcal{B}_k \) and \( \mathcal{B}_k \) of the space \( B_{n_s}(T) \) leads to the following observations.

i) The hierarchical structure of the bases \( \mathcal{B}_k, k \in \mathbb{N} \), can be used for several practices in numerical analysis like compressing data and surfaces.

ii) If we denote by \( T_{\alpha,k} \) (resp. \( T_{\gamma_j,k} \)) the number of B-coefficients of each \( \phi_{i,k}^\alpha, 1 \leq i \leq 3 \), (resp. \( \psi_k^{\gamma_j} \)) that are not necessarily equal to zeros, then by straightforward computation we get

\[
T_{\alpha,k} = \binom{n_s+2}{2} - 2 \left( \frac{\rho_s+2}{2} \right) - \left( \frac{|\alpha|+1}{2} \right) \quad \text{and} \quad T_{\gamma_j,k} = \binom{n_s+2}{2} - 3 \left( \frac{\rho_s+2}{2} \right).
\]

These B-coefficients are solution of linear systems of size \( T_{\alpha,k} \) or \( T_{\gamma_j,k} \), that derive from Hermite interpolation problems given by (5). For the elements of \( \mathcal{B}_k \), the number of B-coefficients is only \( T_{\alpha,s} \) for \( \phi_{i,s}^\alpha \) and \( T_{\gamma_j,s} \) for \( \psi_k^{\gamma_j} \), when \( \alpha \) is such that \( \rho_{s-1} + 1 \leq |\alpha| \leq \rho_s, \gamma_j \in I_s \setminus I_{s-1} \) and \( 0 \leq s \leq k \). Then the size of their corresponding systems are respectively \( T_{\alpha,s} \) and \( T_{\gamma_j,s} \). However, the complexity of determining the basis \( \mathcal{B}_k \) is far less than that of \( \mathcal{B}_k \).

iii) Computation of the polynomial \( p_k \in B_{n_k}(T) \) at several points: According to (ii), each basis function \( \phi_{i,k}^\alpha \) or \( \psi_k^{\gamma_j} \) is determined by a large number of B-coefficients, so the computation of the polynomial \( p_k \) needs at lot of operations. As in practice this computation is required for several points \( T \), we conclude that it is useful to use the new basis which allows us to reduce extensively the number of operations.

3.1 Application

In this section, we are interested in the decomposition of polynomials that arise from some unisolvent interpolation problems.

Lemma 3.5 The data set \( D_k(u) \) given in (3) (resp. in (4)) uniquely determines a SBB-polynomial \( p_k \) of degree \( n_k = 2k \) (resp. \( n_k = 2k + 1 \)) solution of the problem \( H_k \).

Proof The proof is similar to the proof of the bivariate case (see [4]). Indeed, assume that \( p_k \) is written in its SBB-form, and the corresponding Bézier coefficients are numbered as in Figure (1). Assume that \( n_k = 2k \); it is simple to verify that

\[
\dim B_{n_k}(T) = \text{card}(D_k(u)) = \binom{2k+2}{2} = (2k+1)(k+1).
\]
Then, showing that $D_k(u)$ is a determining set for $B_{n_k}(T)$ is equivalent to show that $D_k(u)$ uniquely determines all $B$-coefficients of $p_k$. Indeed, the $C^k$ smoothness at $v_1$ implies that the data set $\{D^\alpha u(v_1), \mid \alpha \mid \leq k - 1\}$ uniquely determines the $\left(\frac{(k-1)+2}{2}\right) = \frac{k(k+1)}{2}$ coefficients corresponding to domain points marked with $\bullet$ closest to vertex $v_1$ (see Figure(1)). The situation at $v_2$ and $v_3$ is analogous. Moreover, it is easy to see that $\{D^\gamma u(B), \mid \gamma \mid \leq k\}$ uniquely determines the $\left(\frac{k+2}{2}\right) = \frac{(k+1)(k+2)}{2}$ coefficients corresponding to domain points marked with $\bullet$ (diamond). Thus, a total of $\frac{3(k+1)}{2}(k+1) + \frac{(k+1)(k+2)}{2} = (2k+1)(k+1)$ coefficients are already determined, and this completes the proof.

\[\Box\]

**Corollary 3.6** Let $p_k \in B_{2k+1}(T)$ (resp. $p_k \in B_{2k}(T)$) be the Hermite spherical polynomial interpolant associated to the data set (4) (resp. (3)). Then $p_k$ and $p_k$ can be decomposed in the form

$$ p_k = p_0 + q_1 + \ldots + q_k, $$

$$ \overline{p}_k = \overline{p}_0 + \overline{q}_1 + \ldots + \overline{q}_k, $$

where $p_0$ is the spherical polynomial interpolating the value of $u$ at $V_i$, $i = 1, 2, 3$ and $\overline{p}_0$ is the spherical polynomial equal to $u(B)$, while

$$ q_s = \sum_{i=1}^{3} \sum_{\mid \alpha \mid = s} D^\alpha (u - p_{s-1})(V_i) \varphi^\alpha_{i,s} + \sum_{\mid \gamma \mid = s-1} D^\gamma (u - p_{s-1})(B_j) \psi^\gamma_s $$

and

$$ \overline{q}_s = \sum_{i=1}^{3} \sum_{\mid \alpha \mid = s-1} D^\alpha (u - \overline{p}_{s-1})(V_i) \widetilde{\varphi}^\alpha_{i,s} + \sum_{\mid \gamma \mid = s} D^\gamma (u - \overline{p}_s)(B_j) \widetilde{\psi}^\gamma_s. $$

For each $1 \leq s \leq k$, the elements $\{\varphi^\alpha_{i,s}, \psi^\gamma_s, \mid \alpha \mid = s, \mid \gamma \mid = s - 1\}$ (resp.$\{\widetilde{\varphi}^\alpha_{i,s}, \widetilde{\psi}^\gamma_s, \mid \alpha \mid = s - 1, \mid \gamma \mid = s\}$) are the last Hermite basis functions for $B_{2s+1}(T)$ (resp. $B_{2s}(T)$).

**Corollary 3.7** The collection $\{\varphi^\alpha_{i,s}, \psi^\gamma_s, \mid \alpha \mid = s, \mid \gamma \mid = s - 1, 1 \leq s \leq k \text{ and } 1 \leq i \leq 3\}$ (resp.$\{\widetilde{\varphi}^\alpha_{i,s}, \widetilde{\psi}^\gamma_s, \mid \alpha \mid = s - 1, \mid \gamma \mid = s, 1 \leq s \leq k \text{ and } 1 \leq i \leq 3\}$) form a basis for $B_{2k+1}(T)$, (resp. $B_{2k}(T)$).

**Proof** This result follows from Theorem 3.4 with $n_k = 2k+1$ (resp. $n_k = 2k$), taking into account that the functions $\widetilde{\varphi}^\alpha_{i,0}$ and $\widetilde{\psi}^\gamma_s$ such that $\mid \alpha \mid = \mid \gamma \mid = -1$ are

Figure 1. $k = 2$, $n_k = 4$. 
According to Remark 2, we have two cases:

Case : \( n_k = 2k + 1 \)
- The explicit expression for \( T_{\alpha,k} \) and \( T_{\gamma,k} \) are
  \[ T_{\alpha,k} = (k + 1)^2 - \frac{|\alpha|(|\alpha| + 1)}{2} \quad \text{and} \quad T_{\gamma,k} = \frac{k(k + 1)}{2}. \]
- The total number of B-coefficients needed for the determination of \( B_{k,T} \) is given by
  \[ \Sigma_{2k+1} = 3 \sum_{|\alpha|\leq k} T_{\alpha,k} + \sum_{|\gamma|\leq k-1} T_{\gamma,k} = 3 + \frac{37}{4} k + \frac{89}{8} k^2 + \frac{25}{4} k^3 + \frac{11}{8} k^4, \]
- The number of B coefficients for the determination of the new basis \( \bar{B}_{k,T} \) is given by
  \[ \sigma_{2k+1} = \sum_{s=0}^{k} \left[ 3 \sum_{|\alpha|=s} T_{\alpha,s} + \sum_{|\gamma|=s-1} T_{\gamma,s} \right] = 3 + \frac{47}{6} k + \frac{15}{2} k^2 + \frac{19}{6} k^3 + \frac{1}{2} k^4. \]

Case : \( n_k = 2k \)
- The explicit expression for \( T_{\alpha,k} \) and \( T_{\gamma,k} \) are
  \[ T_{\alpha,k} = (k + 1)^2 - \frac{|\alpha|(|\alpha| + 1)}{2} \quad \text{and} \quad T_{\gamma,k} = \frac{(k + 1)(k + 2)}{2}. \]
- The total number of B-coefficients needed for the determination of \( B_{k,T} \) is given by
  \[ \Sigma_{2k} = 3 \sum_{|\alpha|\leq k-1} T_{\alpha,k} + \sum_{|\gamma|\leq k} T_{\gamma,k} = 1 + \frac{19}{4} k + \frac{65}{8} k^2 + \frac{23}{4} k^3 + \frac{11}{8} k^4, \]
- The number of B coefficients for the determination of the new basis \( \bar{B}_{k,T} \) is given by
  \[ \sigma_{2k} = \sum_{s=0}^{k} \left[ 3 \sum_{|\alpha|=s} T_{\alpha,s} + \sum_{|\gamma|=s} T_{\gamma,s} \right] = 1 + \frac{29}{6} k + \frac{13}{2} k^2 + \frac{19}{6} k^3 + \frac{1}{2} k^4. \]

In the following table, we give \( \Sigma_{2k+1}, \sigma_{2k+1}, \Sigma_{2k} \) and \( \sigma_{2k} \), for the first values of \( k \).

<table>
<thead>
<tr>
<th>k</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Sigma_{2k+1} )</td>
<td>3</td>
<td>31</td>
<td>138</td>
<td>411</td>
<td>970</td>
<td>1968</td>
<td>3591</td>
<td>6058</td>
<td>9621</td>
<td>14565</td>
<td>21208</td>
</tr>
<tr>
<td>( \sigma_{2k+1} )</td>
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<td>22</td>
<td>82</td>
<td>220</td>
<td>485</td>
<td>938</td>
<td>1652</td>
<td>2712</td>
<td>4215</td>
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<td>8998</td>
</tr>
<tr>
<td>( \Sigma_{2k} )</td>
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<td>21</td>
<td>111</td>
<td>355</td>
<td>870</td>
<td>1806</td>
<td>3346</td>
<td>5706</td>
<td>9135</td>
<td>13915</td>
<td>20361</td>
</tr>
<tr>
<td>( \sigma_{2k} )</td>
<td>1</td>
<td>16</td>
<td>70</td>
<td>200</td>
<td>455</td>
<td>896</td>
<td>1596</td>
<td>2640</td>
<td>4125</td>
<td>6160</td>
<td>8866</td>
</tr>
</tbody>
</table>
In order to illustrate our results, we give in the next section some numerical examples.

4. Numerical Examples

In this section, we give an example which illustrate the theoretical results. Let $T$ be the spherical triangle with vertices $V_1 = (0, \frac{\pi}{2})$, $V_2 = (0, \frac{\pi}{4})$, $V_3 = (\frac{3\pi}{4}, \frac{\pi}{4})$. In this example, we describe the decomposition of the Hermite interpolant $p_2$ to the Sphere-Like Surfaces associated at the function $f(x, y, z) = \sum_{i=1}^{3} (g_i(x, y, z))^{-\frac{1}{2}}$ where $g_i(x, y, z) = \left(\frac{x}{\alpha_i}\right)^2 + \left(\frac{y}{\alpha_{i+1}}\right)^2 + \left(\frac{z}{\alpha_{i+2}}\right)^2$ and $(\alpha_1, \ldots, \alpha_5) = (5, 1, 2, 5, 1)$.

B is the center of gravity of $T$(see Figure (2)). From corollary 3.7, we have $p_2 = p_0 + q_1 + q_2$. In Figure 3 we present the graphs of $p_0$, $p_2$ and the detail functions $q_1$ and $q_2$, for $n_k = 2k + 1$, in this case we have

$$D_0(u) = \{u(V_i), 1 \leq i \leq 3\},$$

$$D_1(u) = \{D_{a_i}^\alpha u(V_i), u(B), \alpha = 0, 1; 1 \leq i, j \leq 3; i \neq j\},$$

and

$$D_2(u) = \{D_{a_i}^\alpha u(V_i), u(B), D_{a_i}u(B), D_{a_j}u(B); \alpha = 0, 1, 2; 1 \leq i, j \leq 3; i \neq j\}.$$

<table>
<thead>
<tr>
<th>$|f_T - p_0|_\infty$</th>
<th>$|f_T - (p_0 + q_1)|_\infty$</th>
<th>$|f_T - p_1|_\infty$</th>
<th>$|f_T - (p_0 + q_1 + q_2)|_\infty$</th>
<th>$|f_T - p_2|_\infty$</th>
</tr>
</thead>
<tbody>
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<td>$5.0113 \times 10^{-3}$</td>
<td>$4.9384 \times 10^{-3}$</td>
<td>$1.5202 \times 10^{-4}$</td>
<td>$1.4375 \times 10^{-4}$</td>
</tr>
</tbody>
</table>
5. Conclusion

In this paper, we proposed to extend the hierarchical bivariate Hermite Interpolant to the spherical case. Let $T$ be an arbitrary spherical triangle of the unit sphere $S$ and let $u$ be a function defined over the triangle $T$. For $k \in \mathbb{N}$, we considered a Hermite spherical Interpolant problem $H_k$ defined by some data scheme $D_k(u)$ and which admits a unique solution $p_k$ in the space $B_{n_k}(T)$ of homogeneous Bernstein-Bézier polynomials of degree $n_k = 2k$ (resp. $n_k = 2k + 1$) defined on $T$. 
References


