

## Discrete-Time $GI/D-MSP/1/K$ Queue with $N$ Threshold Policy

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**Abstract.** This paper presents a discrete-time single-server finite buffer  $N$  threshold policy queue with renewal input and discrete Markovian service process. The server terminates service whenever the system becomes empty, and recommences service as soon as the number of waiting customers in the queue is  $N$ . We obtain the system-length distributions at pre-arrival and arbitrary epochs using the supplementary variable and the imbedded Markov chain techniques. Various performance measures such as the loss probability, mean queue length and mean waiting time in the queue along with some numerical results have been presented. The proposed model has potential applications in the areas of computer and telecommunication systems.

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**Keywords:** Discrete Markovian service process;  $N$  threshold policy; Finite buffer; Queue; Supplementary variable

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## 1. Introduction

Discrete-time queueing systems have received considerable attention due to their wide applications in the performance analysis of communication and telecommunication systems. Their significance has further increased due to the emergence of the broadband integrated services digital network (B-ISDN) which can provide transfer of video, voice and data through high speed local area networks (LANs), on-demand video distribution, and video telephony, etc. Discrete-time queueing models are better fitted than their continuous-time counterparts to determine performance measures in computer and digital telecommunication networks, because of the clock-driven operation of those systems. Extensive discussion on applications

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of discrete-time queues is available in Bruneel and Kim (1993), Woodward (1994) and Takagi (1993).

Queueing models with non-renewal arrival/service processes are generally employed to model networks of complex computer and communication systems to model the correlative and bursty characteristic of traffic streams in high-speed packet based network. Discrete Markovian service process (D-MSP) has been brought in because of the limitations of Bernoulli process in capturing correlation among the service times. The discrete Markovian service process is similar to the Discrete-time Markovian Arrival Process (D-MAP), where arrivals are replaced by service completions. Several queueing systems have been studied by presuming input process as D-MAP such as Chaudhry and Gupta (2003), Liu and Neuts (1994). Alfa et al. (2000) discussed the asymptotic behavior of  $GI/MSP/1$  queue using the perturbation theory. The analysis of finite buffer  $GI/MSP/1$  queue has been presented by Bocharov et al. (2003). Using imbedded Markov chain and semi-Markov process, they derived stationary characteristics of system performance. Shioda (2003) analyzed the departure process of  $MAP/SM/1$  queue. The analysis of  $GI/MSP/1$  queue has been presented in Gupta and Banik (2007) using imbedded Markov chain and supplementary variable techniques for finite buffer system, and the matrix-geometric method and the renewal theory for infinite buffer system. The stationary discrete-time  $GI/D-MSP/1$  queue with finite and infinite buffers has been analyzed in Samanta et al. (2009). Lim et al. (2013) studied infinite buffer  $GI/Geo/1$  queue with  $N$  threshold policy. Sikdar (2012) discussed the  $N$  threshold policy in the finite buffer  $GI/MSP/1$  queue.

In this paper, we consider a discrete-time  $GI/D-MSP/1/K$  queue with  $N$  threshold policy for late-arrival system with delayed access. Lifetime elongation for wireless sensor network using queueing system with  $N$  threshold policy is very useful that minimizes power consumption in sensor node Jiang et al. (2012). The another major application of the  $N$  threshold policy is in manufacturing system. The analysis is based on the use of the supplementary variable technique and the imbedded Markov chain technique. The server turns off when the system is empty, and turns the server on when  $N$  ( $N \geq 1$ ) or more customers are present. When the server is turned off, the server may not work till  $N$  customers are present in the system. We obtain the steady-state system length distributions at pre-arrival and arbitrary epochs. It may be noted that queueing models such as  $GI/Geo/1/K$ ,  $GI/D-PH/1/K$  and  $GI/D-MSP/1/K$  queues are special cases of  $GI/D-MSP/1/K$  queue with  $N$  threshold policy. Moreover, the modeling of discrete-time queue is more involved and quite distinct from the analysis used for the corresponding continuous-time queueing model.

The rest of the paper is organized as follows: Section 2 presents the system description and necessary notations. In Section 3, we obtain the stationary system length probabilities at pre-arrival epochs using imbedded Markov chain technique, and derive a relation between pre-arrival and arbitrary epoch probabilities using the supplementary variable technique. Various performance measures are evaluated in Section 4. Section 5 contains numerical results in the form of tables and graphs to show the effectiveness of the model parameters. Section 6 concludes our paper.

## 2. Description of the model

Let us consider a  $GI/D-MSP/1/K$  queue with  $N$  threshold policy wherein the inter-arrival times  $A$  of two successive arrivals are independent and identically distributed (i.i.d.) random variables with probability mass function (p.m.f.)  $a_k = P(A = k)$ ,  $k \geq 1$ , probability generating function (p.g.f.)  $A(z) = \sum_{k=1}^{\infty} a_k z^k$ ,

$|z| \leq 1$  and mean inter-arrival time  $\lambda^{-1} = A^{(1)}(1)$ , where  $A^{(1)}(1)$  is the first derivative of  $A(z)$  w.r.t.  $z$  at  $z = 1$ .

The departure process of the queueing system is a  $D - MSP$  and is governed by an underlying  $m$ -state Markov chain having probability  $L_{ij}$ ,  $1 \leq i, j \leq m$ , with a transition from state  $i$  to  $j$  without service completion and having probability  $M_{ij}$ ,  $1 \leq i, j \leq m$ , with a transition from state  $i$  to  $j$  with a service completion. Let  $\mathbf{L} = (L_{ij})$ , and  $\mathbf{M} = (M_{ij})$  be the  $m \times m$  non-negative matrices with both having at least one positive entry and  $(\mathbf{L} + \mathbf{M})\mathbf{e} = \mathbf{e}$ , where  $\mathbf{e}$  is the  $m \times 1$  column vector with all elements equal to one. The sum  $(\mathbf{L} + \mathbf{M})$  is a stochastic matrix corresponding to an irreducible Markov chain underlying D-MSP. Let  $\bar{\mathbf{\Pi}}$  be the  $1 \times m$  stationary vector of the underlying Markov chain, i.e.,  $\bar{\mathbf{\Pi}}(\mathbf{L} + \mathbf{M}) = \bar{\mathbf{\Pi}}, \bar{\mathbf{\Pi}}\mathbf{e} = 1$ .

The fundamental service rate of the stationary D-MSP is  $\mu^* = \frac{\mu^*(t)}{t} = \bar{\mathbf{\Pi}}\mathbf{M}\mathbf{e}$ . The customers are served according to D-MSP. The offered load  $\rho$  is defined as  $\rho = \lambda/\mu^*$ . We consider a discrete-time single-server finite buffer queueing system under the Late Arrival System with Delayed Access (LAS-DA). We assume that the time axis is slotted into intervals of equal length with the length of a slot being unity, and is marked by the points  $0, 1, 2, \dots, t, \dots$ . In LAS-DA a potential arrival occurs in  $(t-, t)$  and a potential departure occurs in  $(t, t+)$ . For a detailed description of these concepts, see Hunter (1983) and Gravey and Hébuterne (1992).

The state of the system prior to a potential arrival is described by the following random variables:

- $N_{t-}$  = number of customers present in the system,
- $\xi_{t-} = \{j\}$  where the server is in the  $j$ th ( $1 \leq j \leq m$ ) phase of service process,
- $U_{t-}$  = the remaining inter-arrival time for the next arrival.

For the sake of convenience, we have used the notation  $t$  in place of  $t-$ . Let us define the joint probabilities by

$$\begin{aligned}
 p_{i,j}(u, t) &= P\{N_t = i, \xi_t = j, U_t = u, \text{ the server is turned off}\}, \\
 &0 \leq i \leq N - 1, u \geq 0, \\
 \pi_{i,j}(u, t) &= P\{N_t = i, \xi_t = j, U_t = u, \text{ the server is turned on and working}\}, \\
 &1 \leq i \leq K, u \geq 0.
 \end{aligned}$$

In the steady-state, let us define  $p_{i,j}(u) = \lim_{t \rightarrow \infty} p_{i,j}(u, t)$  and  $\pi_{i,j}(u) = \lim_{t \rightarrow \infty} \pi_{i,j}(u, t)$ . Further, let  $\mathbf{p}_i(u)$  be the row vector of order  $1 \times m$  whose  $j$ -th component  $p_{i,j}(u)$  denotes the probability of  $i$  ( $0 \leq i \leq N - 1$ ) customers in the system, the server is turned off and the service process in phase  $j$  ( $1 \leq j \leq m$ ) and define the vector-generating function  $\mathbf{p}_i^*(z) = \sum_{u=0}^{\infty} \mathbf{p}_i(u)z^u$ ,  $|z| \leq 1$ . Note that  $\mathbf{p}_i = \mathbf{p}_i^*(1)$  is the  $1 \times m$  vector at an arbitrary epoch when server is turned off. Similarly, let  $\mathbf{\Pi}_i(u)$  be the row vector of order  $1 \times m$  whose  $j$ -th component  $\pi_{i,j}(u)$  denotes the probability of  $i$  customers in the system, server turned on and the service process in phase  $j$  and define the vector-generating function  $\mathbf{\Pi}_i^*(z) = \sum_{u=0}^{\infty} \mathbf{\Pi}_i(u)z^u$ ,  $|z| \leq 1$ . Again  $\mathbf{\Pi}_i = \mathbf{\Pi}_i^*(1)$  is the  $1 \times m$  vector at an arbitrary epoch when the server is turned on and working.

### 3. Analysis of the model

In this section, we carry out the system length distributions at pre-arrival and arbitrary epochs using the imbedded Markov chain and supplementary variable techniques.

#### 3.1 System length distribution at pre-arrival epochs

We obtain the system length distribution at pre-arrival epochs. Let  $t_0, t_1, t_2, \dots$  be the time epochs at which successive arrivals occurs and  $t_n^-$  be the time epoch just before the arrival instant  $t_n$ . The inter-arrival times  $T_{n+1} = t_{n+1} - t_n$ ,  $n = 0, 1, 2, \dots$  are i.i.d. random variables with c.d.f.  $A(u)$ . The state of the system at  $t_i^-$  is defined as  $\{N_{t_i^-}, \xi_{t_i^-}\}$ , where  $N_{t_i^-}$  and  $\xi_{t_i^-} = \{j\}$  are the same as defined in Section 2. In the limiting case, we define the following probabilities:

$$p_{i,j}^- = \lim_{t \rightarrow \infty} P\left\{N_{t_i^-} = i, \xi_{t_i^-} = j, \text{ server is turned off}\right\}, 0 \leq i \leq N-1, 1 \leq j \leq m,$$

$$\pi_{i,j}^- = \lim_{t \rightarrow \infty} P\left\{N_{t_i^-} = i, \xi_{t_i^-} = j, \text{ server is turned on and working}\right\}, 1 \leq i \leq K,$$

$$1 \leq j \leq m,$$

where  $p_{i,j}^-$  ( $\pi_{i,j}^-$ ) is the probability that there are  $i$  customers in the system just prior to an arrival of a customer when the server is turned off (turned on and working) and the phase of the service process  $j$ . Let  $\mathbf{p}_i^-$  and  $\mathbf{\Pi}_i^-$  be the row vectors of order  $1 \times m$  whose  $j$ -th components are  $p_{i,j}^-$  and  $\pi_{i,j}^-$ , respectively.

Let  $\mathbf{S}_n^{(k)}$  be the matrix of order  $m \times m$  whose element  $[\mathbf{S}_n^{(k)}]_{ij}$  is the conditional probability that  $n$  customers are served during a time of length  $k$  slots and the service process passes to phase  $j$ , provided initially the service process is in phase  $i$ . Since  $\mathbf{S}_n^{(k)}$  represents that  $n$  customers are served during a time of length  $k$  slots, this means either no customer or one customer is served in the first slot, and  $n$  or  $(n-1)$  customers, respectively, are served in the remaining  $(k-1)$  slots. Combining these arguments, we obtain the following recursive relation:

$$\mathbf{S}_n^{(k)} = \mathbf{L}\mathbf{S}_n^{(k-1)} + \mathbf{M}\mathbf{S}_{n-1}^{(k-1)}, \quad k \geq 1, n \geq 0,$$

with  $\mathbf{S}_0^{(0)} = \mathbf{I}$  and  $\mathbf{S}_{-1}^{(k)} = \mathbf{S}_n^{(k)} = \mathbf{0}$ ,  $n > k \geq 0$ , where  $\mathbf{I}$  and  $\mathbf{0}$  are the identity and zero matrices of order  $m \times m$ , respectively. Let  $\mathbf{S}^{(k)}(z)$  be the matrix-generating function of  $\mathbf{S}_n^{(k)}$ , then

$$\mathbf{S}^{(k)}(z) = \sum_{n=0}^{\infty} \mathbf{S}_n^{(k)} z^n = [\mathbf{S}^{(1)}(z)]^k = [\mathbf{L} + \mathbf{M}z]^k,$$

where  $\mathbf{S}^{(1)}(z) = \mathbf{L} + \mathbf{M}z$  is the matrix-generating function of the number of customers served during a slot. Let  $\mathbf{S}_n$  denotes the matrix of order  $m \times m$  which represents that  $n$  customers complete service during an inter-arrival period  $A$  of a customer. Therefore,

$$\mathbf{S}_n = \sum_{k=1}^{\infty} a_k \mathbf{S}_n^{(k)}, \quad n \geq 0.$$

If  $\mathbf{S}(z)$  is the matrix-generating function of  $\mathbf{S}_n$ , then

$$\mathbf{S}(z) = \sum_{n=0}^{\infty} \mathbf{S}_n z^n = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} a_k \mathbf{S}_n^{(k)} z^n = \sum_{k=1}^{\infty} a_k [\mathbf{L} + \mathbf{M}z]^k.$$

Setting  $z = 1$  in the above equation, we get  $\mathbf{S} \equiv \mathbf{S}(1) = \sum_{k=1}^{\infty} a_k [\mathbf{L} + \mathbf{M}]^k$ . We note that the matrix  $\mathbf{S}$  is stochastic and that the stationary vector  $\bar{\mathbf{\Pi}}$  defined earlier satisfies  $\bar{\mathbf{\Pi}}\mathbf{S} = \bar{\mathbf{\Pi}}$ ,  $\bar{\mathbf{\Pi}}\mathbf{e} = 1$ . The matrix  $\mathbf{S}$  also represents the number of customers served during an inter-arrival time with the phase change of the underlying Markov chain during the inter-arrival time.

Let  $\hat{\mathbf{S}}_r$  denotes the matrix of order  $m \times m$  which represents that at least  $(r + 1)$  customers complete service during an inter-arrival period of a customer. Then

$$\hat{\mathbf{S}}_n = \mathbf{S} - \sum_{r=0}^n \mathbf{S}_r, \quad 0 \leq n \leq K - 1.$$

Let  $\mathfrak{R} = (\mathfrak{R}_{ij})$  be the transition probability matrix with finite state space  $\Omega = \{(n, 0) : 0 \leq n \leq N - 1\} \cup \{(n, 1) : 1 \leq n \leq K\}$  where  $(n, i)$  represents that  $n$  customers in the queue at pre-arrival epoch and  $i = 0(1)$  corresponds to the state of the server, turned off (turned on and busy). Observing the state of the system at two consecutive imbedded points, we have the one step transition probability matrix (TPM)  $\mathfrak{R}$  of the form:

$$\mathfrak{R} = \begin{matrix} & \text{off} & \text{on} \\ \begin{matrix} \text{off} \\ \text{on} \end{matrix} & \begin{pmatrix} \mathbf{A}_{N \times N} & \mathbf{B}_{N \times K} \\ \mathbf{C}_{K \times N} & \mathbf{D}_{K \times K} \end{pmatrix} \end{matrix}$$

Blocks  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$  and  $\mathbf{D}$  refer to the transition from turned off state to turned off state, turned off state to turned on state, turned on state to turned off state and turned on state to turned on state, respectively and are given by the following expression:

$$\mathbf{A}_{i,j} = \begin{cases} \mathbf{I}_m, & 0 \leq i \leq N - 2, 1 \leq j \leq N - 1, i + 1 = j, \\ \hat{\mathbf{S}}_{N-1}, & i = N - 1, j = 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathbf{B}_{i,j} = \begin{cases} \mathbf{S}_{N-j}, & i = N - 1, 1 \leq j \leq N, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathbf{C}_{i,j} = \begin{cases} \hat{\mathbf{S}}_i, & 1 \leq i \leq K - 1, j = 0, \\ \mathbf{C}_{i-1,j}, & i = K, 0 \leq j \leq N - 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathbf{D}_{i,j} = \begin{cases} \mathbf{S}_{i+1-j}, & 1 \leq i \leq K - 1, 1 \leq j \leq K, (i + 1) \geq j, \\ \mathbf{D}_{i-1,j}, & i = K, 1 \leq j \leq K, \\ 0, & \text{otherwise,} \end{cases}$$

The pre-arrival epoch probabilities  $\mathbf{p}_i^-$  ( $0 \leq i \leq N-1$ ) and  $\mathbf{\Pi}_i^-$  ( $1 \leq i \leq K$ ) can be obtained by solving the system of equations  $[\mathbf{p}_0^-, \mathbf{p}_1^-, \dots, \mathbf{p}_{N-1}^-, \mathbf{\Pi}_1^-, \dots, \mathbf{\Pi}_K^-] = [\mathbf{p}_0^-, \mathbf{p}_1^-, \dots, \mathbf{p}_{N-1}^-, \mathbf{\Pi}_1^-, \dots, \mathbf{\Pi}_K^-] \mathfrak{R}$ . To solve it we have used the GTH (Grassmann, Taksar and Heyman) algorithm given in Latouche and Ramaswami (1999, pp. 123).

### 3.2 System length distribution at arbitrary epochs

To obtain the system length distribution at arbitrary epochs we develop the vector-difference equations by observing the state of the system at two consecutive time epochs  $t-$  and  $(t+1)-$  in case of LAS-DA. In the steady-state, we have, for  $u \geq 1$

$$\mathbf{p}_0(u-1) = \mathbf{p}_0(u) + \mathbf{\Pi}_1(u)\mathbf{M}, \quad (1)$$

$$\mathbf{p}_n(u-1) = \mathbf{p}_n(u) + \mathbf{p}_{n-1}(0)a_u, \quad 1 \leq n \leq N-1, \quad (2)$$

$$\mathbf{\Pi}_1(u-1) = \mathbf{\Pi}_1(u)\mathbf{L} + \{\mathbf{\Pi}_2(u) + \mathbf{\Pi}_1(0)a_u\}\mathbf{M}, \quad (3)$$

$$\begin{aligned} \mathbf{\Pi}_n(u-1) &= \{\mathbf{\Pi}_n(u) + \mathbf{\Pi}_{n-1}(0)a_u\}\mathbf{L} + \{\mathbf{\Pi}_{n+1}(u) + \mathbf{\Pi}_n(0)a_u\}\mathbf{M}, \\ &2 \leq n \leq N-2, \end{aligned} \quad (4)$$

$$\begin{aligned} \mathbf{\Pi}_{N-1}(u-1) &= \{\mathbf{\Pi}_{N-1}(u) + \mathbf{\Pi}_{N-2}(0)a_u\}\mathbf{L} + \{\mathbf{\Pi}_N(u) + \mathbf{\Pi}_{N-1}(0)a_u \\ &+ \mathbf{p}_{N-1}(0)a_u\}\mathbf{M}, \end{aligned} \quad (5)$$

$$\begin{aligned} \mathbf{\Pi}_N(u-1) &= \{\mathbf{\Pi}_N(u) + \mathbf{\Pi}_{N-1}(0)a_u + \mathbf{p}_{N-1}(0)a_u\}\mathbf{L} + \mathbf{\Pi}_{N+1}(u)\mathbf{M} \\ &+ \mathbf{\Pi}_N(0)a_u\mathbf{M}, \end{aligned} \quad (6)$$

$$\begin{aligned} \mathbf{\Pi}_n(u-1) &= \{\mathbf{\Pi}_n(u) + \mathbf{\Pi}_{n-1}(0)a_u\}\mathbf{L} + \{\mathbf{\Pi}_{n+1}(u) + \mathbf{\Pi}_n(0)a_u\}\mathbf{M}, \\ &N+1 \leq n \leq K-2, \end{aligned} \quad (7)$$

$$\begin{aligned} \mathbf{\Pi}_{K-1}(u-1) &= \{\mathbf{\Pi}_{K-1}(u) + \mathbf{\Pi}_{K-2}(0)a_u\}\mathbf{L} + \{\mathbf{\Pi}_{K-1}(0) + \mathbf{\Pi}_K(0)\}a_u\mathbf{M} \\ &+ \mathbf{\Pi}_K(u)\mathbf{M}, \end{aligned} \quad (8)$$

$$\mathbf{\Pi}_K(u-1) = \mathbf{\Pi}_K(u)\mathbf{L} + \{\mathbf{\Pi}_{K-1}(0) + \mathbf{\Pi}_K(0)\}a_u\mathbf{L}. \quad (9)$$

The terms  $\mathbf{p}_i(0)$  and  $\mathbf{\Pi}_i(0)$  denote  $1 \times m$  vectors whose  $i$ -th component are the respective rates of entering to that state with remaining inter-arrival time equal to

zero. Multiplying (1) to (9) by  $z^u$  and summing over  $u$  from 1 to  $\infty$ , we obtain

$$z\mathbf{p}_0^*(z) = \mathbf{p}_0^*(z) - \mathbf{p}_0(0) + \{\mathbf{\Pi}_1^*(z) - \mathbf{\Pi}_1(0)\}\mathbf{M}, \tag{10}$$

$$z\mathbf{p}_n^*(z) = \mathbf{p}_n^*(z) + \mathbf{p}_{n-1}(0)A(z) - \mathbf{p}_n(0), \quad 1 \leq n \leq N - 1, \tag{11}$$

$$z\mathbf{\Pi}_1^*(z) = \{\mathbf{\Pi}_1^*(z) - \mathbf{\Pi}_1(0)\}\mathbf{L} + \{\mathbf{\Pi}_2^*(z) - \mathbf{\Pi}_1(0)\}\mathbf{M} + \mathbf{\Pi}_1(0)A(z)\mathbf{M}, \tag{12}$$

$$z\mathbf{\Pi}_n^*(z) = \{\mathbf{\Pi}_n^*(z) + \mathbf{\Pi}_{n-1}(0)A(z) - \mathbf{\Pi}_n(0)\}\mathbf{L} + \{\mathbf{\Pi}_{n+1}^*(z) - \mathbf{\Pi}_{n+1}(0)\}\mathbf{M} \\ + \mathbf{\Pi}_n(0)A(z)\mathbf{M}, \quad 1 \leq n \leq N - 2, \tag{13}$$

$$z\mathbf{\Pi}_{N-1}^*(z) = \{\mathbf{\Pi}_{N-1}^*(z) + \mathbf{\Pi}_{N-2}(0)A(z) - \mathbf{\Pi}_{N-1}(0)\}\mathbf{L} + \mathbf{\Pi}_{N-1}^*(z)\mathbf{M} \\ + \{\mathbf{\Pi}_{N-1}(0) + \mathbf{p}_{N-1}(0)\}A(z)\mathbf{M} - \mathbf{\Pi}_N(0)\mathbf{M}, \tag{14}$$

$$z\mathbf{\Pi}_N^*(z) = \{\mathbf{\Pi}_N^*(z) + \mathbf{\Pi}_{N-1}(0)A(z) + \mathbf{p}_{N-1}(0)A(z) - \mathbf{\Pi}_N(0)\}\mathbf{L} \\ + \{\mathbf{\Pi}_{N+1}^*(z) - \mathbf{\Pi}_{N+1}(0)\}\mathbf{M} + \mathbf{\Pi}_N(0)A(z)\mathbf{M}, \tag{15}$$

$$z\mathbf{\Pi}_n^*(z) = \{\mathbf{\Pi}_n^*(z) + \mathbf{\Pi}_{n-1}(0)A(z) - \mathbf{\Pi}_n(0)\}\mathbf{L} + \{\mathbf{\Pi}_{n+1}^*(z) - \mathbf{\Pi}_{n+1}(0)\} \\ + \mathbf{\Pi}_n(0)A(z)\mathbf{M}, \quad N + 1 \leq n \leq K - 2, \tag{16}$$

$$z\mathbf{\Pi}_{K-1}^*(z) = \{\mathbf{\Pi}_{K-1}^*(z) - \mathbf{\Pi}_{K-1}(0)\}\mathbf{L} + \{\mathbf{\Pi}_{K-1}(0) + \mathbf{\Pi}_K(0)\}A(z)\mathbf{M} \\ + \mathbf{\Pi}_K^*(z)\mathbf{M}, \tag{17}$$

$$z\mathbf{\Pi}_K^*(z) = \{\mathbf{\Pi}_K^*(z) + (\mathbf{\Pi}_{K-1}(0) + \mathbf{\Pi}_K(0))A(z) - \mathbf{\Pi}_K(0)\}\mathbf{L}. \tag{18}$$

LEMMA 3.1 *The mean number of entrances into the system per unit time equals the mean arrival rate, i.e.,*

$$\sum_{n=0}^{N-1} \mathbf{p}_n(0)\mathbf{e} + \sum_{n=1}^K \mathbf{\Pi}_n(0)\mathbf{e} = \lambda. \tag{19}$$

*Proof* Post-multiplying equations (10) to (18) by the vector  $\mathbf{e}$ , adding them, using  $(\mathbf{L} + \mathbf{M})\mathbf{e} = \mathbf{e}$ , we get after simplification

$$\sum_{n=0}^{N-1} \mathbf{p}_n^*(z) + \sum_{n=1}^K \mathbf{\Pi}_n^*(z) = \frac{A(z) - 1}{z - 1} \left\{ \sum_{n=0}^{N-1} \mathbf{p}_n(0) + \sum_{n=1}^K \mathbf{\Pi}_n(0) \right\}.$$

Taking the limit as  $z \rightarrow 1$  and using normalization condition  $\sum_{n=0}^{N-1} \mathbf{p}_n\mathbf{e} + \sum_{n=1}^K \mathbf{\Pi}_n\mathbf{e} = 1$ , we obtain the desired result. ■

### 3.2.1 Relations between system length distributions at arbitrary and pre-arrival epochs

In order to obtain the relation between system length probabilities at arbitrary and pre-arrival epochs, we first connect the pre-arrival epoch probabilities  $\mathbf{p}_n^-$  and  $\mathbf{\Pi}_n^-$  with the rates  $\mathbf{p}_n(0)$  and  $\mathbf{\Pi}_n(0)$ . These are given by

$$\mathbf{p}_n^- = \frac{1}{\lambda}\mathbf{p}_n(0), \quad 0 \leq n \leq N - 1, \quad \text{and} \quad \mathbf{\Pi}_n^- = \frac{1}{\lambda}\mathbf{\Pi}_n(0), \quad 1 \leq n \leq K, \tag{20}$$

where  $\lambda$  is given by (19). Our main objective is to obtain the probabilities of the number of customers in the system at arbitrary epoch when the server is turned off  $\mathbf{p}_n$  ( $0 \leq n \leq N - 1$ ) or server is turned on and working  $\mathbf{\Pi}_n$  ( $1 \leq n \leq K$ ).

The relation between pre-arrival  $\{\mathbf{p}_n^-\}_{n=0}^N$  and arbitrary  $\{\mathbf{\Pi}_n\}_{n=1}^K$  epoch probabilities are given by

$$\mathbf{\Pi}_K = \lambda \mathbf{\Pi}_{K-1,1}^- \mathbf{L}(\mathbf{I} - \mathbf{L})^{-1}, \quad (21)$$

$$\mathbf{\Pi}_{K-1} = \{\mathbf{\Pi}_K \mathbf{M} + \lambda(\mathbf{\Pi}_{K-2}^- - \mathbf{\Pi}_{K-1}^-) \mathbf{L} + \lambda \mathbf{\Pi}_{K-1}^- \mathbf{M}\} (\mathbf{I} - \mathbf{L})^{-1}, \quad (22)$$

$$\begin{aligned} \mathbf{\Pi}_n &= \{\mathbf{\Pi}_{n+1} \mathbf{M} + \lambda(\mathbf{\Pi}_{n-1}^- - \mathbf{\Pi}_n^-) \mathbf{L} + \lambda(\mathbf{\Pi}_n^- - \mathbf{\Pi}_{n+1}^-) \mathbf{M}\} (\mathbf{I} - \mathbf{L})^{-1}, \\ n &= K-2, \dots, N+1, \end{aligned} \quad (23)$$

$$\begin{aligned} \mathbf{\Pi}_N &= \{\mathbf{\Pi}_{N+1} \mathbf{M} + \lambda(\mathbf{p}_{N-1}^- + \mathbf{\Pi}_{N-1}^- - \mathbf{\Pi}_N^-) \mathbf{L}\} (\mathbf{I} - \mathbf{L})^{-1} \\ &+ \{\lambda(\mathbf{\Pi}_N^- - \mathbf{\Pi}_{N+1}^-) \mathbf{M}\} (\mathbf{I} - \mathbf{L})^{-1}, \end{aligned} \quad (24)$$

$$\begin{aligned} \mathbf{\Pi}_{N-1} &= \{\mathbf{\Pi}_N \mathbf{M} + \lambda(\mathbf{\Pi}_{N-2}^- - \mathbf{\Pi}_{N-1}^-) \mathbf{L}\} (\mathbf{I} - \mathbf{L})^{-1} \\ &+ \{\lambda(\mathbf{p}_{N-1}^- + \mathbf{\Pi}_{N-1}^- - \mathbf{\Pi}_N^-) \mathbf{M}\} (\mathbf{I} - \mathbf{L})^{-1}, \end{aligned} \quad (25)$$

$$\begin{aligned} \mathbf{\Pi}_n &= \{\mathbf{\Pi}_{n+1} \mathbf{M} + \lambda(\mathbf{\Pi}_{n-1}^- - \mathbf{\Pi}_n^-) \mathbf{L} + \lambda(\mathbf{\Pi}_n^- - \mathbf{\Pi}_{n+1}^-) \mathbf{M}\} (\mathbf{I} - \mathbf{L})^{-1}, \\ n &= N-2, \dots, 2, \end{aligned} \quad (26)$$

$$\mathbf{\Pi}_1 = \{\mathbf{\Pi}_2 \mathbf{M} + \lambda(\mathbf{\Pi}_1^- - \mathbf{\Pi}_2^-) \mathbf{M} - \lambda \mathbf{\Pi}_1^- \mathbf{L}\} (\mathbf{I} - \mathbf{L})^{-1}, \quad (27)$$

$$\mathbf{p}_n = \mathbf{p}_{n-1}^-, \quad 1 \leq n \leq N-1. \quad (28)$$

Finally, using the normalization condition, we get  $\mathbf{p}_0$  as  $\mathbf{p}_0 = \bar{\mathbf{\Pi}} - (\sum_{n=1}^{N-1} \mathbf{p}_n + \sum_{n=1}^K \mathbf{\Pi}_n)$ .

#### 4. Performance measures

As state probabilities at various epochs are known, performance measures of the queue can be easily obtained. The average number of customers in the system ( $L_s$ ) and the average number of customers in the queue ( $L_q$ ) at an arbitrary epoch are given by

$$L_s = \sum_{n=1}^{N-1} n \mathbf{p}_n \mathbf{e} + \sum_{n=1}^K n \mathbf{\Pi}_n \mathbf{e}, \quad L_q = \sum_{n=1}^{N-1} n \mathbf{p}_n \mathbf{e} + \sum_{n=1}^K (n-1) \mathbf{\Pi}_n \mathbf{e}.$$

Using the Little's formula, the average waiting time in the system ( $W_s$ ) and the average waiting time in the queue ( $W_q$ ), respectively are given by

$$W_s = L_s / \lambda', \quad W_q = L_q / \lambda',$$

where  $\lambda' = \lambda(1 - P_{loss})$  is the effective arrival rate, and  $P_{loss} = \mathbf{\Pi}_K^- \mathbf{e}$  represents the probability of loss or blocking.

#### 5. Numerical results

To demonstrate the applicability of the analytical results obtained in this paper, we present some numerical results in the form of tables and graphs. Numerical results presented in this paper were performed using Mathematica Software and are presented here with five decimal places. In Table 1, the results are given for



Table 1. Distribution of number of customers in the system at various epochs for the *GI/D-PH/1/15* queue with  $N(= 6)$  threshold policy

Pre-arrival		Arbitrary		Pre-arrival		Arbitrary	
$i$	$\pi_i^- e$	$\pi_i e$	$i$	$\pi_i^- e$	$\pi_i e$		
1	0.04536	0.04885	9	0.00404	0.00377		
2	0.06677	0.06539	10	0.00159	0.00149		
3	0.07822	0.07362	11	0.00063	0.00059		
4	0.09043	0.07928	12	0.00025	0.00023		
5	0.09604	0.08366	13	0.00010	0.00009		
6	0.06742	0.06334	14	0.00004	0.00004		
7	0.02611	0.02441	15	0.00001	0.00001		
8	0.01025	0.00957					

$P_0^- e = P_1^- e = P_2^- e = P_3^- e = P_4^- e = P_5^- e = 0.08546$   
 $P_0 e = 0.11838, P_1 e = P_2 e = P_3 e = P_4 e = P_5 e = 0.08546$   
 $L_q = 2.65081, W_q = 13.25420$  and  $P_{loss} = 0.00001$ .

Table 2. Distribution of number of customers in the system at various epochs for the *GI/D-MSP/1/15* queue with  $N(= 6)$  threshold policy

Pre-arrival		Arbitrary		Pre-arrival		Arbitrary	
$i$	$\pi_i^- e$	$\pi_i e$	$i$	$\pi_i^- e$	$\pi_i e$		
1	0.05401	0.04761	9	0.00019	0.00020		
2	0.06662	0.05843	10	0.00003	0.00004		
3	0.06573	0.06020	11	0.00001	0.00001		
4	0.07805	0.06254	12	0.00000	0.00000		
5	0.07189	0.06319	13	0.00000	0.00000		
6	0.03168	0.03343	14	0.00000	0.00000		
7	0.00573	0.00604	15	0.00000	0.00000		
8	0.00103	0.00109					

$P_0^- e = P_1^- e = P_2^- e = P_3^- e = P_4^- e = P_5^- e = 0.10417$   
 $P_0 e = 0.14638, P_1 e = P_2 e = P_3 e = P_4 e = P_5 e = 0.10417$   
 $L_q = 2.39477, W_q = 11.9738$  and  $P_{loss} = 0.00000$ .

*GI/D-PH/1/15* queue with  $N = 6$  for arbitrary inter-arrival time distribution:  $a_2 = 0.6, a_6 = 0.3, a_{20} = 0.1, \lambda = 0.2$ . The service time distribution is a discrete phase type renewal process with representation  $(\alpha, T)$  where  $\alpha = (0.3, 0.7)$  and  $T = \begin{pmatrix} 0.6 & 0.074 \\ 0.0575 & 0.45 \end{pmatrix}$  so that  $\mu = 0.47454$  and  $\rho = 0.42146$ . For the same inter-arrival distribution, similar results have been presented in Table 2 for the *GI/D-MSP/1/15* queue with  $N = 6$ . For this table and for all the figures, the *D - MSP* representation is taken as  $L = \begin{pmatrix} 0.35 & 0.1 \\ 0.12 & 0.3 \end{pmatrix} M = \begin{pmatrix} 0.3 & 0.25 \\ 0.2 & 0.38 \end{pmatrix}$  so that  $\mu = 0.565672, \rho = 0.35356$ .

Figure 1 depicts the system length probabilities that the server is turned on and working for different  $N$  threshold values. Observation is that the probabilities are increasing up to  $N - 1$  and then has a sharp decrease in the trend indicating that after  $N - 1$  threshold value, the server becomes busy immediately thereby decrease in the probabilities. Further indication is that as  $N$  value increases, there is a certain drop in the level of the busyness probabilities which certainly makes the model more effective from server point of view.

The effect of arrival rate ( $\lambda$ ) on the average queue length ( $L_q$ ) for different  $N$

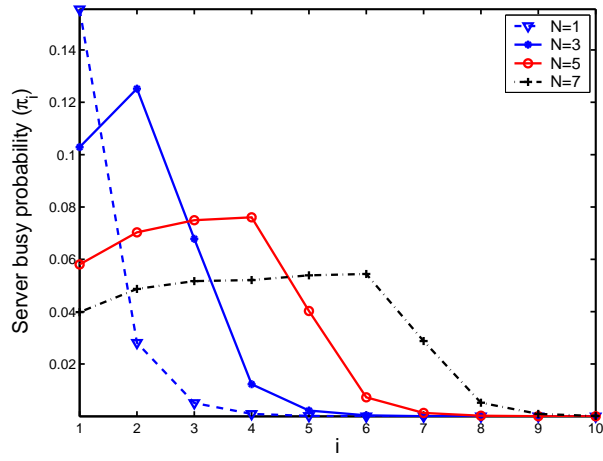


Figure 1. System length distributions when server turned on and working.

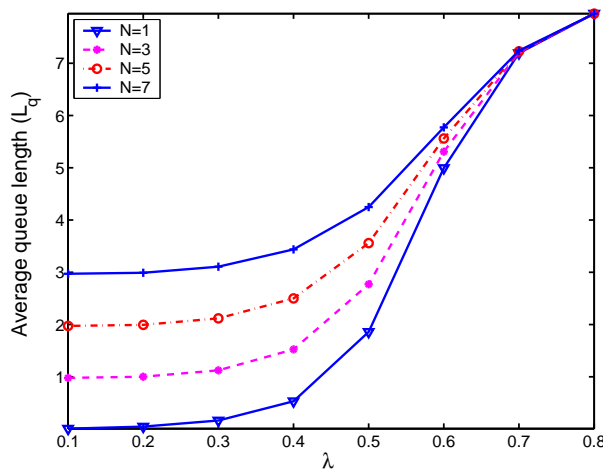


Figure 2. Impact of arrival rate on average queue length.

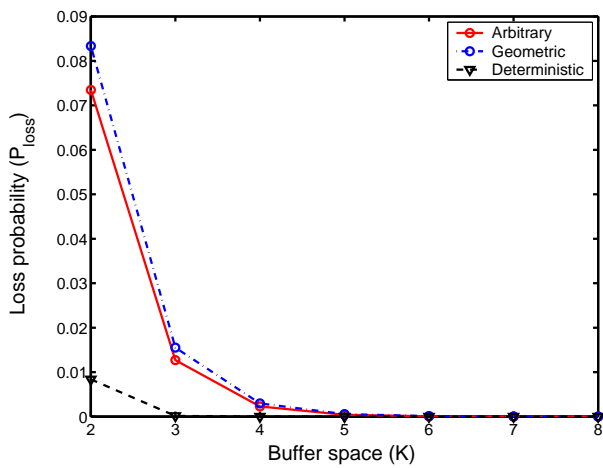


Figure 3. Effect of buffer space on loss probability.

threshold values is shown in Figure 2 for geometric arrival distribution. Here we observe that as  $\lambda$  increases obviously there is an increase in the queue length. From customers' point of view, certainly one has to wait longer in the queue when the arrival rate is slower and starting threshold value ( $N$ ) is larger. Thus from Figures 1 and 2, one can see that proper choice of  $N$  is necessary to keep the customers' waiting and the server's utilization are at an optimum level.

The effect of the buffer space ( $K$ ) on the loss probability is shown in Figure 3 for different arrival distributions Geometric, Arbitrary and Deterministic. It can be seen that as  $K$  increases, there is a sharp decrease in the loss probability and finally tends to zero value as larger  $K$  occurs in the case of infinite buffer space models. Further, amongst all distributions considered here, the Deterministic distribution yields least loss probability.

## 6. Conclusion

In this paper, we have carried out an analysis of a discrete-time single server  $N$  threshold policy in the finite buffer queue with renewal input and discrete Markovian service process. The proposed model has potential applications in the areas of computer and telecommunication systems. We have obtained the steady-state distributions of system length at pre-arrival and arbitrary epochs using the supplementary variable and the imbedded Markov chain techniques. Various performance measures such as the probability of blocking, average queue-length and average waiting time in the queue have been carried out. The results for the early arrival system can also be obtained in a similar manner. The techniques used in this paper can be applied to analyze more complex models such as  $GI/D-MSP/c/N$  and  $DMAP/D-MSP/1/N$  queues with  $N$  threshold policy. We may study these queueing models under single or multiple or any other type of vacation policy which are left for future investigations.

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