Certain Sufficient Conditions for Close-to-Convexity of Analytic Functions

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Abstract. The object of this paper is to derive certain sufficient conditions for close-to-convexity of certain analytic functions defined on the unit disk \( \Delta := \{ z \in \mathbb{C} : |z| < 1 \} \).

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1. Introduction

Let \( \mathcal{H}(\Delta) \) be the class of analytic functions in the unit disk \( \Delta := \{ z \in \mathbb{C} : |z| < 1 \} \) and \( \mathcal{H}[a, n] \) be the subclass of functions of the form \( f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \). We denote \( \mathcal{H} = \mathcal{H}[1, 1] \). Let \( \mathcal{A} \) denote the subclass of \( \mathcal{H} \) normalized by the conditions \( f(0) = 0 = f'(0) - 1 \). Thus, the class \( \mathcal{A} \) consists of the functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n.
\]

Let \( \mathcal{S} \) be the subclass of \( \mathcal{A} \) consisting of univalent functions.

A function \( p(z) = 1 + p_1 z + p_2 z^2 + \cdots \) is said to be in the class \( \mathcal{P} \) if \( \text{Re} \, p(z) > 0 \). For two analytic functions \( f \) and \( g \), we say that \( f \) is subordinate to \( g \) or \( g \) is superordinate to \( f \), denoted by \( f \prec g \), if there is a Schwarz function \( w \) with \( |w(z)| \leq |z| \) such that \( f(z) = g(w(z)) \). If \( g \) is univalent, then \( f \prec g \) if and only if \( f(0) = g(0) \) and \( f(\Delta) \subseteq g(\Delta) \). A function \( f \in \mathcal{A} \) is starlike if \( f(\Delta) \) is starlike domain with respect to 0, and a function \( f \in \mathcal{A} \) is convex if \( f(\Delta) \) is a convex domain. Analytically, the
prerequisites are equivalent to the following conditions \( \frac{zf'(z)}{f(z)} \in \mathcal{P} \) and \( 1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P} \), respectively. The class of starlike and convex functions of order \( \alpha, (0 \leq \alpha < 1) \) is defined as follows:

\[
\Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha
\]

and

\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha.
\]

These classes are denoted by \( S^*(\alpha) \) and \( K(\alpha) \) respectively. The class of close to convex functions is defined by

\[
\mathcal{C}(\alpha) := \{ f : f \in \mathcal{A}; \ \Re \left( \frac{f'(z)}{g'(z)} \right) > \alpha, z \in \delta, 0 \leq \alpha < 1; g \in \mathcal{K} \}.
\]

It is well known [1] that \( f \in K(\alpha) \Leftrightarrow zf'(z) \in S^*(\alpha) \). Thus, if \( f \in S^*(\alpha) \), then \( f \in \mathcal{C}(\alpha) \).

The following Lemma is needed in the present investigation:

**Lemma 1.1** [2, 3] Let the function \( w(z) \) be analytic in \( \Delta \) with \( w(0) = 0 \). If \( |w(z)| \) attains its maximum value on the circle \( |z| < 1 \) at a point \( z_0 \in \Delta \), then \( z_0 w'(z_0) = cw(z_0) \), where \( c \geq 1 \).

2. **Main Results**

**Theorem 2.1** Let \( c \geq 1 \) and one of the following conditions holds

1. \( A = 1, 0 < B < 1 \)
2. \( 0 < A < 1, 0 \leq B < A \).

If the function \( f \in \mathcal{A} \) satisfies the inequality

\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 1 + \frac{Ac}{1 + A} + \frac{(1 + A)Bc}{(1 + B)^2} (z \in \Delta),
\]

then

\[
|f'(z) - 1| < |A - Bf'(z)|.
\]

**Proof** Let the function \( w \) be defined as

\[
f'(z) = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad w(z) \neq -\frac{1}{B}.
\]

Then, clearly \( w \) is analytic in the unit disk \( \Delta \) with \( w(0) = 0 \). From (2), by a simple computation, we get

\[
1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{Azw'(z)}{(1 + Aw(z))} - \frac{Bzw'(z)}{(1 + Bw(z))}.
\]
Suppose that there is a point \( z_0 \) in the unit disk \( \Delta \) with the properties \( |w(z_0)| = 1 \) and \( |w(z)| < 1 \), whenever \( |z| < |z_0| \). Now, from the Lemma 1.1, we have

\[
    z_0 w(z_0) = cw(z_0), \quad (c \geq 1, w(z_0) = e^{i\theta}, \theta \in \mathbb{R}).
\]

From (3) and (4), we obtain

\[
    \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) = 1 + \frac{Ac(\cos(\theta) + A)}{1 + A^2 + 2A \cos(\theta)} - \frac{Bc(\cos(\theta) + B)}{1 + B^2 + 2B \cos(\theta)} := u(\theta).
\]

A simple calculation shows that \( u(\theta) \) attains its maximum at \( \theta = 0 \) and

\[
    \max_{\theta \in \mathbb{R}} \{u(\theta)\} = 1 + \frac{Ac}{1 + A} + \frac{(1 + A)Bc}{(1 + B)^2}.
\]

Which is a contradiction to our hypothesis. Thus, \( |w(z)| < 1, \ z \in \Delta \) which implies that \( |f'(z) - 1| < |A - Bf'(z)| \). This completes the proof.

If we set \( B = 0 \) in the Theorem 2.1, then we have:

**Corollary 2.2** Let \( c \geq 1 \) and \( 0 < A < 1 \). If the function \( f \in A \) satisfies the inequality

\[
    \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 1 + \frac{Ac}{1 + A} \quad (z \in \Delta),
\]

then

\[
    \text{Re}(f'(z)) > 1 - A.
\]

Which equivalently can be written as \( f \in C(1 - A) \).

If we set \( A = 1/2 \) and \( c = 1 \) in the Corollary 2.2, then we have:

**Corollary 2.3** If the function \( f \in A \) satisfies the inequality

\[
    \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 2.33 \quad (z \in \Delta),
\]

then

\[
    \text{Re}(f'(z)) > 1/2.
\]

Which equivalently can be written as \( f \in C(1/2) \).

Setting \( A = 1 \) in the Theorem 2.1, we have:

**Corollary 2.4** Let \( c \geq 1 \) and \( 0 < B < 1 \). If the function \( f \in A \) satisfies the inequality

\[
    \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 1 + \frac{c}{2} + \frac{2Bc}{(1 + B)^2} \quad (z \in \Delta),
\]

then

\[
    |f'(z)| < \frac{2}{1 - B}.
\]
Setting \( B = 1/2 \) and \( c = 2 \) in the above corollary, we have:

**Corollary 2.5**  If the function \( f \in A \) satisfies the inequality

\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 2.88 \quad (z \in \Delta),
\]

then

\[ |f'(z)| < 4. \]

3. **Conclusion**

In this paper several sufficient conditions for close-to-convexity of analytic functions are obtained. Further this paper leaves a scope to the researchers to discuss more general results in this direction using differential subordination.

4. **Acknowledgement**

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**References**