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A Non-Markovian Batch Arrival Queue with Service Interruption & Extended Server Vacation

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Abstract. A single server provides service to all arriving customers with service time following general distribution. After every service completion the server has the option to leave for phase one vacation of random length with probability p or continue to stay in the system with probability $1 - p$. As soon as the completion of phase one vacation, the server may take phase two vacation with probability q or to remain in the system with probability $1 - q$, after phase two vacation again the server has the option to take phase three vacation with probability r or to remain in the system with probability $1-r$. The vacation times are assumed to be general. The server is interrupted at random and the duration of attending interruption follows exponential distribution. Also we assume, the customer whose service is interrupted goes back to the head of the queue where the arrivals are Poisson. The time dependent probability generating functions have been obtained in terms of their Laplace transforms and the corresponding steady state results have been obtained explicitly. Also the mean number of customers in the queue and system and the waiting time in the queue and system are also derived. Particular cases and numerical results are discussed.

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1. Introduction

Queueing system are powerful tool for modeling communication networks, transportation networks, production lines, operating systems, etc. In recent years, computer networks and data communication systems are the fastest growing technologies, which lead to glorious development in many applications. For example, the swift advance in internet, audio data traffic, video data traffic, etc.

Vacation queues have been the subject of deep study in recent years because of their theoretical structure as well as their applicability in various real life situations. Recently the $M^{[X]}/G/1$ queue with vacation has drawn the attention of various researchers notable among them are Baba (1986, 1987), Lee (1989), Choudhury and Madan (2005), Madan and Al-Rawwash (2005), Deepak Gupta et al. (2011), Badamchi Zadeh (2012) and Kalyanaraman and Suvitha (2012).

Queuing systems with server vacations and/or random system breakdowns have been studied by numerous researchers including the survey of Doshi (1986), Choi and Park (1990), Takine and Sengupta (1997), Wang et al. (2001), Tian and Zhang (2006), Fuhrmann (1981), Cramer (1988), Borthakur and Chaudhury (1997), Chaudhury (2000), Badamchi Zadeh and Shankar (2008). White and Christie (1985) have studied queues with service interruptions. They consider an M/M/1 queueing system with exponentially distributed interruptions. General distributed service times and interruptions are considered by Thiruvengadam (1963), Baskar et al. (2011), Balamani (2012), Maragatha Sundari and Srinivasan (2012), Vijaya Laxmi and Seleshi (2013).

We assume that the customers arrive to the service station in batches of variable size, but are served one by one. We assume that the service times, vacation times have a general distribution while the time to interruptions is exponentially distributed.

In this paper, we consider $M^{[X]}/G/1$ queueing system with service interruption, after every service completion the server has the option to leave for phase one vacation of random length with probability p or continue to stay in the system with probability $1-p$. As soon as the completion of phase one vacation, the server may take phase two vacation with probability q or to continue staying in the system with probability $1-q$, after phase two vacation again the server has the option to take phase three vacation with probability r or to remain in the system with probability $1-r$. The vacation period has three heterogeneous phases. Also we assume, the customer whose service is interrupted goes back to the head of the queue where the arrivals are Poisson.

Here we derive time dependent probability generating functions in terms of Laplace transforms. We also derive the average queue size, system size and average waiting time in the queue, the system. Some particular cases and numerical results are also discussed.

This paper is organized as follows. The mathematical description of our model is given in section 2. Definitions and Equations governing the system are given in section 3. The time dependent solution have been obtained in section 4 and corresponding steady state results have been derived explicitly in section 5. Average queue size and average waiting time are computed in section 6 and 7 respectively. Particular cases and numerical results are discussed in section 8 and 9 respectively.

2. Mathematical Description of the Model

We assume the following to describe the queueing model of our study.

a) Customers arrive at the system in batches of variable size in a compound

Poisson process and they are provided one by one service on a first come - first served basis. Let $\lambda c_i dt$ $(i = 1, 2, \ldots)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t + dt]$, where $0 \leq c_i \leq 1$ and $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the arrival rate of batches.
b) A single server provides service to all arriving customer, with the service time having general distribution. Let $B(v)$ and $b(v)$ be the distribution and the density function of the service time respectively.
c) We assume interruptions arrive at random while serving the customers and assumed to occur according to a Poisson process with mean rate $\alpha > 0$. Let β be the server rate of attending interruption. Further we assume that once the interruption arrives the customer whose service is interrupted comes back to the head of the queue. Let $\mu(x)dx$ be the conditional probability of completion of the service during the interval $(x, x + dx]$ given that the elapsed service time is x, so that

$$\mu(x) = \frac{b(x)}{1 - B(x)},$$

and therefore,

$$b(s) = \mu(s)e^{-\int_0^s \mu(x)dx},$$

d) After each service is over, the server may take a vacation with probability p or to continue to stay in the system with probability $1 - p$. As soon as the completion of phase one vacation, the server may take phase two vacation with probability q or continue to stay in the system with probability $1 - q$, after phase two vacation again the server has the option to take phase three vacation with probability r or to remain in the system with probability $1 - r$.
e) The server's vacation time follows a general (arbitrary) distribution with distribution function $V_i(t)$ and density function $v_i(t)$. Let $\gamma_i(x)dx$ be the conditional probability of a completion of a vacation during the interval $(x, x + dx]$ given that the elapsed vacation time is x, so that

$$\gamma_i(x) = \frac{v_i(x)}{1 - V_i(x)}, \quad i = 1, 2, 3$$

and therefore,

$$v_i(t) = \gamma_i(t)e^{-\int_0^t \gamma_i(x)dx}, \ i = 1, 2, 3.$$

f) On returning from vacation the server instantly starts serving the customer at the head of the queue if any.
g) Various stochastic processes involved in the system are assumed to be independent of each other.

3. Definitions and Equations Governing the System

We define

$P_n(x, t)$ = Probability that at time t, the server is active providing service and there are n $(n \geq 0)$ customers in the queue excluding the one being served and the elapsed service time for this customer is x. Consequently $P_n(t) = \int_0^{\infty} P_n(x, t)dx$

denotes the probability that at time t there are n customers in the queue excluding one customer in the service irrespective of the value of x.

$V_n^{(i)}(x,t)$ = Probability that at time t, the server is under i^{th} vacation with elapsed vacation time x and there are n $(n \geq 0)$ customers in the queue. Consequently $V_n^{(i)}(t)= \int_0^{\infty} V_n^{(i)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is under i^{th} vacation irrespective of the x for i = 1, 2, 3.

$R_n(t)$= Probability that at time t, the server is inactive due to the arrival of interruption.

$Q(t)$ = Probability that at time t, there are no customers in the queue or in service and the server is idle but available in the system.

According to the mathematical model mentioned above, the system has the following set of differential-difference equations:

$$\frac{\partial}{\partial x}P_0(x,t) + \frac{\partial}{\partial t}P_0(x,t) + [\lambda + \alpha + \mu(x)]P_0(x,t) = 0 \tag{1}$$

$$\frac{\partial}{\partial x}P_n(x,t) + \frac{\partial}{\partial t}P_n(x,t) + [\lambda + \alpha + \mu(x)]P_n(x,t) = \lambda\sum_{k=1}^{n} c_k P_{n-k}(x,t), \quad n \geq 1 \tag{2}$$

$$\frac{\partial}{\partial x}V_0^{(1)}(x,t) + \frac{\partial}{\partial t}V_0^{(1)}(x,t) + [\lambda + \gamma_1(x)]V_0^{(1)}(x,t) = 0 \tag{3}$$

$$\frac{\partial}{\partial x}V_n^{(1)}(x,t) + \frac{\partial}{\partial t}V_n^{(1)}(x,t) + [\lambda + \gamma_1(x)]V_n^{(1)}(x,t) = \lambda\sum_{k=1}^{n} c_k V_{n-k}^{(1)}(x,t), \; n \geq 1 \tag{4}$$

$$\frac{\partial}{\partial x}V_0^{(2)}(x,t) + \frac{\partial}{\partial t}V_0^{(2)}(x,t) + [\lambda + \gamma_2(x)]V_0^{(2)}(x,t) = 0 \tag{5}$$

$$\frac{\partial}{\partial x}V_n^{(2)}(x,t) + \frac{\partial}{\partial t}V_n^{(2)}(x,t) + [\lambda + \gamma_2(x)]V_n^{(2)}(x,t) = \lambda\sum_{k=1}^{n} c_k V_{n-k}^{(2)}(x,t), \; n \geq 1 \tag{6}$$

$$\frac{\partial}{\partial x}V_0^{(3)}(x,t) + \frac{\partial}{\partial t}V_0^{(3)}(x,t) + [\lambda + \gamma_3(x)]V_0^{(3)}(x,t) = 0 \tag{7}$$

$$\frac{\partial}{\partial x}V_n^{(3)}(x,t) + \frac{\partial}{\partial t}V_n^{(3)}(x,t) + [\lambda + \gamma_3(x)]V_n^{(3)}(x,t) = \lambda\sum_{k=1}^{n} c_k V_{n-k}^{(3)}(x,t), \quad n \geq 1 \tag{8}$$

$$\frac{d}{dt}R_0(t) = -(\lambda + \beta)R_0(t) \tag{9}$$

$$\frac{d}{dt}R_n(t) = -(\lambda + \beta)R_n(t) + \lambda\sum_{k=1}^{n} c_k R_{n-k}(t) + \alpha\int_0^{\infty} P_{n-1}(x,t)dx \tag{10}$$

$$\begin{aligned}\frac{d}{dt}Q(t) = & -\lambda Q(t) + \beta R_0(t) + (1-p)\int_0^{\infty}\mu(x)P_0(x,t)dx \\ & + (1-q)\int_0^{\infty}\gamma_1(x)V_0^{(1)}(x,t)dx + (1-r)\int_0^{\infty}\gamma_2(x)V_0^{(2)}(x,t)dx \\ & + \int_0^{\infty}\gamma_3(x)V_0^{(3)}(x,t)dx \end{aligned} \tag{11}$$

Equations are to be solved subject to the following boundary conditions:

$$\begin{aligned} P_n(0,t) =&\lambda c_{n+1}Q(t) + (1-p)\int_0^\infty \mu(x)P_{n+1}(x,t)dx \\ &+ \beta R_{n+1}(t) + (1-q)\int_0^\infty \gamma_1(x)V^{(1)}_{n+1}(x,t)dx \\ &+ (1-r)\int_0^\infty \gamma_2(x)V^{(2)}_{n+1}(x,t)dx \\ &+ \int_0^\infty \gamma_3(x)V^{(3)}_{n+1}(x,t)dx \end{aligned} \tag{12}$$

$$V^{(1)}_n(0,t) = p\int_0^\infty \mu(x)P_n(x,t)dx, \quad n \geq 0 \tag{13}$$

$$V^{(2)}_n(0,t) = q\int_0^\infty \gamma_1(x)\bar{V}^{(1)}_n(x,t)dx, \quad n \geq 0 \tag{14}$$

$$V^{(3)}_n(0,t) = r\int_0^\infty \gamma_2(x)\bar{V}^{(2)}_n(x,t)dx, \quad n \geq 0 \tag{15}$$

we assume that initially there are no customers in the system and the server is idle. So the initial conditions are

$$\begin{aligned} Q(0) =&1,\ V^{(i)}_0(0) = V^{(i)}_n(0) = 0, \quad R_n(0) = 0, \\ P_n(0) =&0 \quad for\ n \geq 0, \quad i = 1,2,3. \end{aligned} \tag{16}$$

4. Generating Functions of the Queue Length: The Time-Dependent Solution

In this section we obtain the transient solution for the above set of differential-difference equations.

Theorem 4.1 *The system of differential difference equations to describe an $M^{[X]}/G/1$ queue with service with service interruption and three phases of vacation are given by equations (1) to (15) with initial conditions (16) and the generating functions of transient solution are given by equation (63) to (67).*

Proof :We define the probability generating functions for i=1, 2, 3.

$$\begin{aligned} P(x,z,t) =&\sum_{n=0}^{\infty} z^n P_n(x,t); P(z,t) = \sum_{n=0}^{\infty} z^n P_n(t) \\ R(z,t) =&\sum_{n=0}^{\infty} z^n R_n(t); C(z) = \sum_{n=1}^{\infty} c_n z^n \\ V^{(i)}(x,z,t) =&\sum_{n=0}^{\infty} z^n V^{(i)}_n(x,t); V^{(i)}(z,t) = \sum_{n=0}^{\infty} z^n V^{(i)}_n(t) \end{aligned} \tag{17}$$

which are convergent inside the circle given by $\mid z \mid \leq 1$ and define the Laplace transform of a function $f(t)$ as

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt, \quad \Re(s) > 0.$$

We take the Laplace transform of equations (1) to (15) and using (16), we obtain

$$\frac{\partial}{\partial x}\bar{P}_0(x,s) + (s+\lambda+\alpha+\mu(x))\bar{P}_0(x,s) = 0 \tag{18}$$

$$\frac{\partial}{\partial x}\bar{P}_n(x,s) + (s+\lambda+\alpha+\mu(x))\bar{P}_n(x,s) = \lambda\sum_{k=1}^{n} c_k\bar{P}_{n-k}(x,s),\ n \geq 1 \tag{19}$$

$$\frac{\partial}{\partial x}\bar{V}_0^{(1)}(x,s) + (s+\lambda+\gamma_1(x))\bar{V}_0^{(1)}(x,s) = 0 \tag{20}$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(1)}(x,s) + (s+\lambda+\gamma_1(x))\bar{V}_n^{(1)}(x,s) = \lambda\sum_{k=1}^{n} c_k\bar{V}_{n-k}^{(1)}(x,s),\ n \geq 1 \tag{21}$$

$$\frac{\partial}{\partial x}\bar{V}_0^{(2)}(x,s) + (s+\lambda+\gamma_2(x))\bar{V}_0^{(2)}(x,s) = 0 \tag{22}$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(2)}(x,s) + (s+\lambda+\gamma_2(x))\bar{V}_n^{(2)}(x,s) = \lambda\sum_{k=1}^{n} c_k\bar{V}_{n-k}^{(2)}(x,s),\ n \geq 1 \tag{23}$$

$$\frac{\partial}{\partial x}\bar{V}_0^{(3)}(x,s) + (s+\lambda+\gamma_3(x))\bar{V}_0^{(3)}(x,s) = 0 \tag{24}$$

$$\frac{\partial}{\partial x}\bar{V}_n^{(3)}(x,s) + (s+\lambda+\gamma_3(x))\bar{V}_n^{(3)}(x,s) = \lambda\sum_{k=1}^{n} c_k\bar{V}_{n-k}^{(3)}(x,s),\ n \geq 1 \tag{25}$$

$$(s+\lambda+\beta)\bar{R}_0(s) = 0 \tag{26}$$

$$(s+\lambda+\beta)\bar{R}_n(s) = \lambda\sum_{k=1}^{n} c_k\bar{R}_{n-k}(s) + \alpha\int_0^\infty \bar{P}_{n-1}(x,s)dx,\ n \geq 1 \tag{27}$$

$$\begin{aligned}(s+\lambda)\bar{Q}(s) = &1 + \beta\bar{R}_0(s) + (1-p)\int_0^\infty \mu(x)P_0(x,s)dx\\ &+ (1-q)\int_0^\infty \gamma_1(x)V_0^{(1)}(x,s)dx\\ &+ (1-r)\int_0^\infty \gamma_2(x)V_0^{(2)}(x,s)dx\\ &+ \int_0^\infty \gamma_3(x)V_0^{(3)}(x,s)dx\end{aligned} \tag{28}$$

$$
\begin{aligned}
P_n(0,s) =& \lambda c_{n+1}\bar{Q}(s) + (1-p)\int_0^\infty \mu(x)P_{n+1}(x,s)dx \\
& + \beta R_{n+1}(s) + (1-q)\int_0^\infty \gamma_1(x)V_{n+1}^{(1)}(x,s)dx \\
& + (1-r)\int_0^\infty \gamma_2(x)V_{n+1}^{(2)}(x,s)dx \\
& + \int_0^\infty \gamma_3(x)V_{n+1}^{(3)}(x,s)dx
\end{aligned} \tag{29}
$$

$$
\bar{V}_n^{(1)}(0,s) = p\int_0^\infty \bar{P}_n(x,s)\mu(x)dx, \quad n \geq 0 \tag{30}
$$

$$
\bar{V}_n^{(2)}(0,s) = q\int_0^\infty \bar{V}_n^{(1)}(x,s)\gamma_1(x)dx, \quad n \geq 0 \tag{31}
$$

$$
\bar{V}_n^{(3)}(0,s) = r\int_0^\infty \bar{V}_n^{(2)}(x,s)\gamma_2(x)dx, \quad n \geq 0. \tag{32}
$$

Now multiplying equations (19), (21), (23), (25), (27) by z^n and summing over n from 1 to ∞, adding to equations (18), (20), (22), (24), (26) and using the generating functions defined in equations (16), we get

$$
\frac{\partial}{\partial x}\bar{P}_n(x,z,s) + [s + \lambda - \lambda C(z) + \alpha + \mu(x)]\bar{P}(x,z,s) = 0 \tag{33}
$$

$$
\frac{\partial}{\partial x}\bar{V}_n^{(1)}(x,z,s) + [s + \lambda - \lambda C(z) + \gamma_1(x)]\bar{V}^{(1)}(x,z,s) = 0 \tag{34}
$$

$$
\frac{\partial}{\partial x}\bar{V}_n^{(2)}(x,z,s) + [s + \lambda - \lambda C(z) + \gamma_2(x)]\bar{V}^{(2)}(x,z,s) = 0 \tag{35}
$$

$$
\frac{\partial}{\partial x}\bar{V}_n^{(3)}(x,z,s) + [s + \lambda - \lambda C(z) + \gamma_3(x)]\bar{V}^{(3)}(x,z,s) = 0 \tag{36}
$$

$$
(s + \lambda - \lambda C(z) + \beta)\bar{R}(z,s) = \alpha z\int_0^\infty \bar{P}(x,z,s)dx \tag{37}
$$

For the boundary conditions, we multiply both sides of equation (28) by z^n sum over n from 0 to ∞ and use the equations (16), we get

$$\begin{aligned} z\bar{P}(0,z,s) =& \lambda C(z)\bar{Q}(s) + \beta\bar{R}(z,s) - \beta\bar{R}_0(s) \\ & + (1-p)\int_0^\infty \mu(x)\bar{P}(x,z,s)dx \\ & - (1-p)\int_0^\infty \mu(x)\bar{P}_0(x,s)dx \\ & + (1-q)\int_0^\infty \gamma_1(x)\bar{V}^{(1)}(x,z,s)dx \\ & - (1-q)\int_0^\infty \gamma_1(x)\bar{V}_0^{(1)}(x,s)dx \\ & + (1-r)\int_0^\infty \gamma_2(x)\bar{V}^{(2)}(x,z,s)dx \\ & - (1-r)\int_0^\infty \gamma_2(x)\bar{V}_0^{(2)}(x,s)dx \\ & + \int_0^\infty \gamma_3(x)\bar{V}_0^{(3)}(x,z,s)dx - \int_0^\infty \gamma_3(x)\bar{V}_0(x,s)dx \end{aligned}$$

Using equation (28), the above equation becomes

$$\begin{aligned} z\bar{P}(0,z,s) =& [1 - s\bar{Q}(s)] + \lambda(C(z)-1)\bar{Q}(s) + \beta\bar{R}(z,s) \\ & + (1-p)\int_0^\infty \mu(x)\bar{P}(x,z,s)dx \\ & + (1-q)\int_0^\infty \gamma_1(x)\bar{V}^{(1)}(x,z,s)dx \\ & + (1-r)\int_0^\infty \gamma_2(x)\bar{V}^{(2)}(x,z,s)dx + \int_0^\infty \gamma_3(x)\bar{V}^{(3)}(x,z,s)dx \end{aligned} \quad (38)$$

Performing similar operation on equations (30) to (32), we get

$$\bar{V}^{(1)}(0,z,s) = p\int_0^\infty \mu(x)\bar{P}(x,z,s)dx \quad (39)$$

$$\bar{V}^{(2)}(0,z,s) = q\int_0^\infty \gamma_1(x)\bar{V}^{(1)}(x,z,s)dx \quad (40)$$

$$\bar{V}^{(3)}(0,z,s) = r\int_0^\infty \gamma_2(x)\bar{V}^{(2)}(x,z,s)dx \quad (41)$$

Integrating equation (33) between 0 to x, we get

$$\bar{P}(x,z,s) = \bar{P}(0,z,s)e^{-[s+\lambda-\lambda C(z))+\alpha]x-\int_0^x \mu(t)dt} \quad (42)$$

where $\bar{P}(0,z,s)$ is given by equation (38). Again integrating equation (42) by parts with respect to x, yields

$$\bar{P}(z,s) = \bar{P}(0,z,s)\left[\frac{1-\bar{B}(s+\lambda-\lambda C(z)+\alpha)}{s+\lambda-\lambda C(z)+\alpha}\right] \tag{43}$$

where

$$\bar{B}(s+\lambda-\lambda C(z)+\alpha) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)+\alpha]x}dB(x)$$

is the Laplace-Stieltjes transform of the essential service time $B(x)$.

Now multiplying both sides of equation (42) by $\mu(x)$ and integrating over x, we obtain

$$\int_0^\infty \bar{P}(x,z,s)\mu(x)dx = \bar{P}(0,z,s)\bar{B}[s+\lambda-\lambda C(z)+\alpha] \tag{44}$$

Similarly, on integrating equations (34) to (36) from 0 to x, we get

$$\bar{V}^{(1)}(x,z,s) = \bar{V}^{(1)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int_0^x \gamma_1(t)dt} \tag{45}$$

$$\bar{V}^{(2)}(x,z,s) = \bar{V}^{(2)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int_0^x \gamma_2(t)dt} \tag{46}$$

$$\bar{V}^{(3)}(x,z,s) = \bar{V}^{(3)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int_0^x \gamma_3(t)dt} \tag{47}$$

where $\bar{V}^{(1)}(0,z,s)$, $\bar{V}^{(2)}(0,z,s)$, and $\bar{V}^{(3)}(0,z,s)$ are given by equations (39) to (41).

Again integrating equations (45) to (47) by parts with respect to x, yields

$$\bar{V}^{(1)}(z,s) = \bar{V}^{(1)}(0,z,s)\left[\frac{1-\bar{V}_1(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{48}$$

$$\bar{V}^{(2)}(z,s) = \bar{V}^{(2)}(0,z,s)\left[\frac{1-\bar{V}_2(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{49}$$

$$\bar{V}^{(3)}(z,s) = \bar{V}^{(3)}(0,z,s)\left[\frac{1-\bar{V}_3(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{50}$$

where

$$\bar{V}_1(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x}dV_1(x)$$

$$\bar{V}_2(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x}dV_2(x)$$

$$\bar{V}_3(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x} dV_3(x)$$

is the Laplace-Stieltjes transform of the first phase, second phase and third phase of vacation time $V_1(x)$, $V_2(x)$ and $V_3(x)$ respectively.

Now multiplying both sides of equations (45), (46), (47) by $\gamma_1(x)$, $\gamma_2(x)$ and $\gamma_3(x)$ and integrating over x, we obtain

$$\int_0^\infty \bar{V}^{(1)}(x,z,s)\gamma_1(x)dx = \bar{V}^{(1)}(0,z,s)\bar{V}_1[s+\lambda-\lambda C(z)] \tag{51}$$

$$\int_0^\infty \bar{V}^{(2)}(x,z,s)\gamma_2(x)dx = \bar{V}^{(2)}(0,z,s)\bar{V}_2[s+\lambda-\lambda C(z)] \tag{52}$$

$$\int_0^\infty \bar{V}^{(3)}(x,z,s)\gamma_3(x)dx = \bar{V}^{(3)}(0,z,s)\bar{V}_3[s+\lambda-\lambda C(z)] \tag{53}$$

Using equation (44) in equation (39), we get

$$\bar{V}^{(1)}(0,z,s) = p\bar{B}(a)\bar{P}(0,z,s) \tag{54}$$

where $a = s+\lambda-\lambda C(z)+\alpha$, $a_1 = s+\lambda-\lambda C(z)$.

Now using equations (51) and (54) in (40), we get

$$\bar{V}^{(2)}(0,z,s) = pq\bar{V}_1(a_1)\bar{B}(a)\bar{P}(0,z,s) \tag{55}$$

By using equations (52) and (55) in (41), we get

$$\bar{V}^{(3)}(0,z,s) = pqr\bar{V}_1(a_1)\bar{V}_2(a_1)\bar{B}(a)\bar{P}(0,z,s) \tag{56}$$

Using equation (44), (51) to (56) in (38), we get

$$[z-\bar{B}(a)(1-p+p\bar{V}_1(a_1)(1-q+q\bar{V}_2(a_1)(1-r+r\bar{V}_3(a_1)))]\bar{P}(0,z,s)$$

$$= [1-s\bar{Q}(s)] + \lambda(C(z)-1)\bar{Q}(s) + \beta\bar{R}(z,s) \tag{57}$$

From (37), we get

$$\bar{R}(z,s) = \frac{\alpha z}{a_2}\bar{P}(0,z,s)\left[\frac{1-\bar{B}(a)}{a}\right] \tag{58}$$

where Now using equation (58) in (57), we have

$$\bar{P}(0,z,s) = \frac{a_2 a[(1-s\bar{Q}(s)) + \lambda(C(z)-1)\bar{Q}(s)]}{aa_2[z-\bar{B}(a)(1-p+p\bar{V}_1(a)a_3] - \alpha z\beta(1-\bar{B}(a))} \tag{59}$$

Similarly using equation (59), in equations (54), (55) and (56), we get

$$\bar{V}^{(1)}(0,z,s) = \frac{p\bar{B}(a)a_2 a[(1-s\bar{Q}(s)) + \lambda(C(z)-1)\bar{Q}(s)]}{aa_2[z-\bar{B}(a)(1-p+p\bar{V}_1(a)a_3] - \alpha z\beta(1-\bar{B}(a))} \tag{60}$$

$$\bar{V}^{(2)}(0,z,s) = \frac{pq\bar{V}_1(a_1)\bar{B}(a)a_2a}{aa_2[z-\bar{B}(a)(1-p+p\bar{V}_1(a)a_3]-\alpha z\beta(1-\bar{B}(a))} [(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)] \tag{61}$$

$$\bar{V}^{(3)}(0,z,s) = \frac{pqr\bar{V}_1(a_1)\bar{V}_2(a_1)\bar{B}(a)a_2a}{aa_2[z-\bar{B}(a)(1-p+p\bar{V}_1(a)a_3)]-\alpha z\beta(1-\bar{B}(a))} [(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)] \tag{62}$$

Using equations (59) to (62) in equations (43), (48), (49), (50) and (58), we get

$$\bar{P}(z,s) = \frac{a_2(1-\bar{B}(a))[(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)]}{aa_2[z-\bar{B}(a)(1-p+p\bar{V}_1(a)a_3)]-\alpha z\beta} \tag{63}$$

$$\bar{V}^{(1)}(z,s) = \frac{p\bar{B}(a)aa_2[1-s\bar{Q}(s)+\lambda(C(z)-1)\bar{Q}(s)]}{aa_2[z-\bar{B}(a)(1-p+p\bar{V}_1(a)a_3)]-\alpha z\beta(1-\bar{B}(a))}\left[\frac{1-V_1(a_1)}{a_1}\right] \tag{64}$$

$$\bar{V}^{(2)}(z,s) = \frac{pq\bar{B}(a)aa_2\bar{V}_1(a_1)[1-s\bar{Q}(s)+\lambda(C(z)-1)\bar{Q}(s)]}{aa_2[z-\bar{B}(a)(1-p+p\bar{V}_1(a)a_3)]-\alpha z\beta(1-\bar{B}(a))}\left[\frac{1-V_2(a_1)}{a_1}\right] \tag{65}$$

$$\bar{V}^{(3)}(z,s) = \frac{pqr\bar{B}(a)aa_2\bar{V}_1(a_1)\bar{V}_2(a_1)[(1-s\bar{Q}(s))+\lambda(C(z)-1)\bar{Q}(s)]}{aa_2[z-\bar{B}(a)(1-p+p\bar{V}_1(a)a_3)]-\alpha z\beta(1-\bar{B}(a))} \times\left[\frac{1-V_3(a_1)}{a_1}\right] \tag{66}$$

$$\bar{R}(z,s) = \frac{\alpha z(1-\bar{B}(a))[1-s\bar{Q}(s)+\lambda(C(z)-1)\bar{Q}(s)]}{aa_2[z-\bar{B}(a)(1-p+p\bar{V}_1(a)a_3)]-\alpha z\beta(1-\bar{B}(a))} \tag{67}$$

Thus $\bar{P}(z,s)$, $\bar{V}^{(1)}(z,s)$, $\bar{V}^{(2)}(z,s)$, $\bar{V}^{(3)}(z,s)$ and $\bar{R}(z,s)$ are completely determined from equations (63) to (67) which completes the proof of the theorem.

5. The Steady State Results

In this section, we shall derive the steady state probability distribution for our queueing model. These probabilities are obtained by suppressing the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the Tauberian property,

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t)$$

In order to determine $\bar{P}(z,s)$, $\bar{V}^{(1)}(z,s)$, $\bar{V}^{(2)}(z,s)$, $\bar{V}^{(3)}(z,s)$ and $\bar{R}(z,s)$ completely, we have yet to determine the unknown Q which appears in the numerators of the right hand sides of equations (63) to (67). For that purpose, we shall use the normalizing condition

$$P(1)+V^{(1)}(1)+V^{(2)}(1)+V^{(3)}(1)+R(1)+Q=1$$

The steady state probabilities for an $M^{[X]}/G/1$ queue with service interruption and three phases of vacation are given by

$$P(1) = \frac{\lambda E(I)\beta[1-\bar{B}(\alpha)]Q}{Dr}$$
$$V^{(1)}(1) = \frac{\lambda p\alpha\beta E(I)\bar{B}(\alpha)E(V_1)Q}{Dr}$$
$$V^{(2)}(1) = \frac{\lambda pq\alpha\beta E(I)\bar{B}(\alpha)E(V_2)Q}{Dr}$$
$$V^{(3)}(1) = \frac{\lambda pqr\alpha\beta E(I)\bar{B}(\alpha)E(V_3)Q}{Dr}$$
$$R(1) = \frac{\lambda\alpha E(I)[1-\bar{B}(\alpha)]Q}{Dr}$$

where

$$Dr = -\lambda E(I)(\alpha+\beta)[1-\bar{B}(\alpha)] + \alpha\beta\bar{B}(\alpha)[1-\lambda pE(I)(E(V_1+q(E(V_2+rE(V_3)))] \quad (68)$$

$P(1),\ V^{(1)}(1),\ V^{(2)}(1), V^{(3)}(1),\ R(1)$ and Q are the steady state probabilities that the server is providing essential service, first phase of vacation, second phase of vacation, third phase of vacation and server under idle respectively without regard to the number of customers in the queue.

Multiplying both sides of equations (63) to (67) by s, taking limit as $s \to 0$, applying Tauberian property and simplifying, we obtain

$$P(z) = \frac{f_1(z)(1-\bar{B})\lambda(C(z)-1)Q}{D(z)} \quad (69)$$
$$V^{(1)}(z) = \frac{pf_1(z)f_2(z)\bar{B}[\bar{V}_1-1]Q}{D(z)} \quad (70)$$
$$V^{(2)}(z) = \frac{pqf_1(z)f_2(z)\bar{V}_1\bar{B}[\bar{V}_2-1]Q}{D(z)} \quad (71)$$
$$V^{(3)}(z) = \frac{pqrf_1(z)f_2(z)\bar{V}_1\bar{V}_2\bar{B}[\bar{V}_3-1]Q}{D(z)} \quad (72)$$
$$R(z) = \frac{\lambda\alpha z(1-\bar{B})(C(z)-1)Q}{D(z)} \quad (73)$$

where

$$D(z) = f_1(z)f_2(z)[z-\bar{B}(1-p+p\bar{V}_1f_3(z))] - \alpha z\beta(1-\bar{B}),$$

$f_1(z) = \lambda - \lambda C(z) + \beta,\ f_2(z) = \lambda - \lambda C(z) + \alpha,\ f_3(z) = 1-q+q\bar{V}_2f_4(z)$ $f_4(z) = 1-r+r\bar{V}_3,\ \bar{B} = \bar{B}(f_2(z)),\ \bar{V}_1 = \bar{V}_1(\lambda-\lambda C(z)),\ \bar{V}_2 = \bar{V}_2(\lambda-\lambda C(z))$ and $\bar{V}_3 = \bar{V}_3(\lambda-\lambda C(z))$.

Let $W_q(z)$ denote the probability generating function of the queue size irrespective of the state of the system. Then adding equations (69) to (73), we obtain

$$W_q(z) = P(z) + V^{(1)}(z) + V^{(2)}(z) + V^{(3)}(z)$$

$$\begin{aligned} W_q(z) =& \frac{f_1(z)(1-\bar{B})\lambda(C(z)-1)Q}{dr} \\ &+ \frac{pf_1(z)f_2(z)\bar{B}[\bar{V}_1-1]Q}{dr} \\ &+ \frac{pqf_1(z)f_2(z)\bar{V}_1\bar{B}[\bar{V}_2-1]Q}{dr} \\ &+ \frac{pqrf_1(z)f_2(z)\bar{V}_1\bar{V}_2\bar{B}[\bar{V}_3-1]Q}{dr} \\ &+ \frac{\lambda\alpha z(1-\bar{B})(C(z)-1)Q}{dr} \end{aligned} \tag{74}$$

we see that for z=1, $W_q(1)$ is indeterminate of the form 0/0. Therefore, we apply L'Hopital's rule and on simplifying we obtain the result (74), where C(1)= 1, $C'(1) = E(I)$ is mean batch size of the arriving customers, $-\bar{B}'(0) = E(B), -\bar{V}_i'(0) = E(V_i)$, $i = 1, 2, 3$.

$$W_q(1) = \frac{\lambda E(I)[(\alpha+\beta)(1-\bar{B}(\alpha)) + p\alpha\beta\bar{B}(\alpha)E(V_1) + q(E(V_2) + rE(V_3))]Q}{Dr} \tag{75}$$

and Dr is given by equation (68). Therefore adding Q to equation (75), equating to 1 and simplifying, we get

$$Q = 1 - \rho \tag{76}$$

and hence the utilization factor ρ of the system is given by

$$\rho = \lambda pE(I)[E(V_1) + q(E(V_2) + rE(V_3))] - \frac{\lambda E(I)}{\bar{B}(\alpha)}(\frac{1}{\beta} + \frac{1}{\alpha})[1 - \bar{B}(\alpha)] \tag{77}$$

where $\rho < 1$ is the stability condition under which the steady state exists. Equation (76) gives the probability that the server is idle. Substituting Q from (76) into (74), we have completely and explicitly determined $W_q(z)$, the probability generating function of the queue size.

6. The Average Queue Size

Let L_q denote the average number of customers in the queue under the steady state. Then

$$L_q = \frac{d}{dz}W_q(z) \;\; at\ z = 1$$

since this formula gives 0/0 form, then we write $W_q(z)$ given in (74) as $W_q(z) = \frac{N(z)}{D(z)}$ where $N(z)$ and $D(z)$ are numerator and denominator of the right hand side of (74) respectively. Then we use

$$L_q = \lim_{z\to 1}\frac{d}{dz}W_q(z) = \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2}\right]Q \tag{78}$$

where primes and double primes in (78) denote first and second derivative at $z = 1$ respectively. Carrying out the derivative at $z = 1$, we have

$$\begin{aligned}
N'(1) =& \lambda E(I)(\alpha+\beta)[1-\bar{B}(\alpha)] \\
&+ \lambda p\alpha\beta E(I)\bar{B}(\alpha)[E(V_1)+q(E(V_2)+rE(V_3))] \qquad (79)\\
N''(1) =& \lambda E(I(I-1))(\alpha+\beta)[1-\bar{B}(\alpha)] \\
&+ 2\lambda^2(E(I))^2(\alpha+\beta)\bar{B}'(\alpha) \\
&+ 2\lambda E(I)(1-\bar{B}(\alpha))[\alpha-\lambda E(I)] \\
&- 2\lambda^2 p(E(I))^2[(\alpha+\beta)\bar{B}(\alpha) \\
&+ \alpha\beta\bar{B}'(\alpha)][E(V_1)+q(E(V_2)+rE(V_3))] \\
&+ \alpha\beta p\bar{B}(\alpha)[\lambda^2(E(I))^2(E(V_1^2)+qE(V_2^2)+rE(V_3^2))) \\
&+ \lambda E(I(I-1))(E(V_1)+q(E(V_2)+rE(V_3))) \\
&+ 2q\lambda^2(E(I))^2E(V_1)(E(V_2)+rE(V_3)) \\
&+ 2qr\lambda^2(E(I))^2E(V_2)E(V_3)] \qquad (80)\\
D'(1) =& \alpha\beta\bar{B}(\alpha)[1-\lambda E(I)p(E(V_1)+q(E(V_2)+rE(V_3)))] \\
&- \lambda E(I)(\alpha+\beta)[1-\bar{B}(\alpha)] \qquad (81)\\
D''(1) =& -2\lambda E(I)(\alpha+\beta)[1+\lambda E(I)\bar{B}'(\alpha)(1-p+p\bar{V}_1f_3(z)) \\
&- \bar{B}(\alpha)(\lambda pE(I)E(V_1)+\lambda pqE(I)(E(V_2)+rE(V_3)))] \\
&+ [2\lambda^2(E(I))^2-\lambda\alpha E(I(I-1))-\lambda\beta E(I(I-1))] \\
&\times [1-\bar{B}(\alpha)(1-p+p\bar{V}_1f_3(z)] \\
&+ \alpha\beta 2\lambda E(I)\bar{B}'(\alpha)(\lambda pE(I)E(V_1)+\lambda pqE(I)(E(V_2)+rE(V_3))) \\
&- \alpha\beta(1-p+p\bar{V}_1f_3(z)[\lambda^2(E(I))^2\bar{B}''(\alpha)-\lambda E(I(I-1))\bar{B}'(\alpha)] \\
&- \alpha\beta\bar{B}(\alpha)[p\lambda^2(E(I))^2(E(V_1^2)+q(E(V_2^2) \\
&+ rE(V_3^2)))+\lambda pE(I(I-1))(E(V_1)+q(E(V_2)+rE(V_3))) \\
&+ 2pq\lambda^2(E(I))^2E(V_1)(E(V_2)+rE(V_3)) \\
&+ 2pqr\lambda^2(E(I))^2E(V_2)E(V_3)]-2\lambda\alpha\beta E(I)\bar{B}'(\alpha) \\
&- \alpha\beta[-\lambda^2(E(I))^2\bar{B}''(\alpha)+\lambda E(I(I-1))\bar{B}'(\alpha)] \qquad (82)
\end{aligned}$$

where $E(B^2)$, $E(V_1^2)$, $E(V_2^2)$, $E(V_3^2)$ are the second moment of service time and vacation times respectively. $E(I(I-1))$ is the second factorial moment of the batch size of arriving customers. Then if we substitute the values $N'(1), N''(1), D'(1), D''(1)$ from equations (79) to (82) into equations (78) we obtain L_q in the closed form. Further, we find the mean system size L using Little's formula. Thus we have

$$L = L_q + \rho \qquad (83)$$

where L_q has been found by equation (78) and ρ is obtained from equation (77).

7. The Average Waiting Time

Let W_q and W denote the mean waiting time in the queue and in the system respectively. Then using Little's formula, we obtain

$$W_q = \frac{L_q}{\lambda}$$

$$W = \frac{L}{\lambda}$$

Where L_q and L have been found in equations (78) and (83).

8. Particular Cases

Case 1: If there is no third phase of extended vacation. i.e, r=0.

Then our model reduces to a single server $M^{[X]}/G/1$ queue with service interruption and two phases of server vacation.

In this case, we find the idle probability Q, utilization factor ρ and the average queue size L_q can be simplified to the following expressions.

$$\begin{aligned} Q =& 1 - \lambda p E(I)[E(V_1) + qE(V_2)] - \frac{\lambda E(I)}{\bar{B}(\alpha)}(\frac{1}{\beta} + \frac{1}{\alpha})[1 - \bar{B}(\alpha)] \\ \rho =& \lambda p E(I)[E(V_1) + qE(V_2)] + \frac{\lambda E(I)}{\bar{B}(\alpha)}(\frac{1}{\beta} + \frac{1}{\alpha})[1 - \bar{B}(\alpha)] \\ L_q =& \left[\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2}\right] Q \end{aligned}$$

where

$$
\begin{aligned}
N'(1) =& \lambda E(I)(\alpha+\beta)[1-\bar{B}(\alpha)] \\
&+ \lambda p\alpha\beta E(I)\bar{B}(\alpha)[E(V_1)+qE(V_2)] \\
N''(1) =& \lambda E(I(I-1))(\alpha+\beta)[1-\bar{B}(\alpha)] \\
&+ 2\lambda^2(E(I))^2(\alpha+\beta)\bar{B}'(\alpha) \\
&+ 2\lambda E(I)(1-\bar{B}(\alpha))[\alpha-\lambda E(I)] \\
&- 2\lambda^2 p(E(I))^2[(\alpha+\beta)\bar{B}(\alpha) \\
&+ \alpha\beta\bar{B}'(\alpha)][E(V_1)+qE(V_2)] \\
&+ \alpha\beta p\bar{B}(\alpha)[\lambda^2(E(I))^2(E(V_1^2)+qEV_2^2)) \\
&+ \lambda E(I(I-1))(E(V_1)+qE(V_2)) \\
&+ 2q\lambda^2(E(I))^2E(V_1)E(V_2) \\
D'(1) =& \alpha\beta\bar{B}(\alpha)[1-\lambda E(I)p(E(V_1)+qE(V_2)] \\
&- \lambda E(I)(\alpha+\beta)[1-\bar{B}(\alpha)] \\
D''(1) =& -2\lambda E(I)(\alpha+\beta)[1+\lambda E(I)\bar{B}'(\alpha)-\bar{B}(\alpha)(\lambda pE(I)E(V_1) \\
&+ \lambda pqE(I)E(V_2))] + [2\lambda^2(E(I))^2 \\
&- \lambda\alpha E(I(I-1)) - \lambda\beta E(I(I-1))][1-\bar{B}(\alpha)] \\
&+ 2\alpha\beta\lambda E(I)\bar{B}'(\alpha)(\lambda pE(I)E(V_1)+\lambda pqE(I)E(V_2)) \\
&- \alpha\beta[\lambda^2(E(I))^2\bar{B}''(\alpha)-\lambda E(I(I-1))\bar{B}'(\alpha)] \\
&- \alpha\beta\bar{B}(\alpha)[p\lambda^2(E(I))^2(E(V_1^2)+qE(V_2^2) \\
&+ \lambda pE(I(I-1))(E(V_1)+qE(V_2)) \\
&+ 2pq\lambda^2(E(I))^2E(V_1)E(V_2)] - 2\lambda\alpha\beta E(I)\bar{B}'(\alpha) \\
&- \alpha\beta[\lambda^2(E(I))^2\bar{B}''(\alpha)+\lambda E(I(I-1))\bar{B}'(\alpha)]
\end{aligned}
$$

Case 2: If there is no second phase and third phase of extended vacation, $C(z) = z$ i.e, $q = r= 0$, $E(I) = 1$ and $E(I(I-1)) = 0$.

Then our model reduces to a single server M/G/1 queue with service interruption and Bernoulli schedule server vacation.

In this case we find the idle probability Q, utilization factor ρ and the average queue size L_q can be simplified to the following expressions.

$$
\begin{aligned}
Q =& 1-\lambda pE(V_1)-\frac{\lambda}{\bar{B}(\alpha)}(\frac{1}{\beta}+\frac{1}{\alpha})[1-\bar{B}(\alpha)] \\
\rho =& \lambda pE(V_1)+\frac{\lambda}{\bar{B}(\alpha)}(\frac{1}{\beta}+\frac{1}{\alpha})[1-\bar{B}(\alpha)] \\
L_q =& \left[\frac{D'(1)N''(1)-N'(1)D''(1)}{2(D'(1))^2}\right]Q
\end{aligned}
$$

where

$$N'(1) = \lambda(\alpha+\beta)[1-\bar{B}(\alpha)] + \lambda p\alpha\beta\bar{B}(\alpha)E(V_1)$$
$$N''(1) = 2\lambda^2(\alpha+\beta)\bar{B}'(\alpha) + 2\lambda(1-\bar{B}(\alpha))[\alpha-\lambda]$$
$$- 2\lambda^2 p[(\alpha+\beta)\bar{B}(\alpha) + \alpha\beta\bar{B}'(\alpha)]E(V_1)$$
$$+ \alpha\beta p\bar{B}(\alpha)\lambda^2 E(V_1^2)$$
$$D'(1) = \alpha\beta\bar{B}(\alpha)(1-\lambda pE(V_1)) - \lambda(\alpha+\beta)[1-\bar{B}(\alpha)]$$
$$D''(1) = -2\lambda(\alpha+\beta)[1+\lambda\bar{B}'(\alpha) - \bar{B}(\alpha)\lambda pE(V_1)]$$
$$+ 2\lambda^2[1-\bar{B}(\alpha)] + 2p\alpha\beta\lambda^2\bar{B}'(\alpha)E(V_1)$$
$$- \alpha\beta\lambda^2\bar{B}''(\alpha) - \alpha\beta\bar{B}(\alpha)p\lambda^2 E(V_1^2)$$
$$- 2\lambda\alpha\beta\bar{B}'(\alpha) - \alpha\beta\lambda^2\bar{B}''(\alpha)$$

The above equations coincides with Balamani (2012).

Case 3: When the vacation follows exponential distribution for case 2 then the results coincide with Baskar et al. (2011).

9. Numerical Results

To numerically illustrate the results obtained in this work, we consider that the service time and vacation times are exponentially distributed with rates μ and γ.

In order to see the effect of various parameters on server's idle time Q, utilization factor ρ and various other queue characteristics such as L, W, L_q, W_q. We base our numerical example on the result found in case 2.

For this purpose in Table 1, we can choose the following arbitrary values: α= 2, β= 4, μ =8, γ =3, p=0.7 while λ varies from 0.1 to 1.0 such that the stability condition is satisfied.

It clearly shows as long as increasing the arrival rate, the server's idle time decreases while the utilization factor, the average queue size, system size and the average waiting time in the queue and the system of our queueing model are all increases.

Table 1. Computed values of various queue characteristics

λ	Q	ρ	L_q	L	W_q	W
0.1	0.957917	0.042083	0.008058	0.050141	0.080576	0.501409
0.2	0.915833	0.084167	0.020099	0.104266	0.100497	0.521330
0.3	0.873750	0.126250	0.036759	0.163009	0.122529	0.543362
0.4	0.831667	0.168333	0.058797	0.227131	0.146993	0.567827
0.5	0.789583	0.210417	0.087139	0.297556	0.174279	0.595112
0.6	0.747500	0.252500	0.122917	0.375417	0.204862	0.625695
0.7	0.705417	0.294583	0.167533	0.462116	0.239332	0.660166
0.8	0.663333	0.336667	0.222744	0.559411	0.278430	0.699264
0.9	0.621250	0.378750	0.290787	0.669537	0.323097	0.743930
1.0	0.579167	0.420833	0.374544	0.795378	0.374544	0.795378

In Table 2, we choose the following values: α= 6, β= 5, μ =7, λ =0.7, p=0.3 while γ varies from 1 to 10 such that the stability condition is satisfied.

It clearly shows as long as increasing the vacation rate, the server's idle time increases while the utilization factor, average queue size, system size and average waiting time in the queue and system of our queueing model are all decreases.

Table 2. Computed values of various queue characteristics

γ	ρ	Q	L_q	L	W_q	W
1	0.430000	0.570000	0.577708	1.007708	0.825297	1.439583
2	0.325000	0.675000	0.308051	0.633051	0.440073	0.904359
3	0.290000	0.710000	0.258894	0.548894	0.369849	0.784135
4	0.272500	0.727500	0.240299	0.512799	0.343285	0.732570
5	0.262000	0.738000	0.230893	0.492893	0.329848	0.704133
6	0.255000	0.745000	0.225318	0.480318	0.321883	0.686169
7	0.250000	0.750000	0.221666	0.471666	0.316666	0.673809
8	0.246200	0.753750	0.219104	0.465354	0.313006	0.664792
9	0.243300	0.756667	0.217215	0.460548	0.310307	0.657926
10	0.241000	0.759000	0.215767	0.456767	0.308239	0.652525

10. Conclusion

In this paper we considered a single server queue with Extended Bernoulli vacation and service interruption. Customers arrive at the system in batches of variable size in a compound Poisson process and the single server provides services to all arriving customers. Using supplementary variable technique the probability generating functions of number of customers in the queue at different server states are obtained. Some performance measures are calculated from the probability generating functions. Further we performed numerical analysis by assuming particular values to the parameters.

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