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B-Spline Method for Two-Point Boundary Value Problems

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Abstract. In this work the collocation method based on quartic B-spline is developed and applied to two-point boundary value problem in ordinary differential equations. The error analysis and convergence of presented method is discussed. The method illustrated by two test examples which verify that the presented method is applicable and considerable accurate.

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1. Introduction

Consider the general form of linear two-point boundary value problem:

$$Ly \equiv y''(x) + p(x)y'(x) + q(x)y(x) = r(x), x \in [a, b], \tag{1}$$

with boundary conditions

$$y(a) = \alpha, y(b) = \beta. \tag{2}$$

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The problem has a unique solution, if $p, q, r \subset C^1[a, b]$ and $q(x) < 0$ [5]. Generally speaking this problem is difficult analytically. Some of the most frequently used numerical methods are shooting, finite difference, finite element and finite volume methods [1],[8] and, etc [2],[7],[13],[16]. These methods, although requiring little computational time, evaluate the approximated solutions only at the collocation points, $y(x_i)$ for $i = 0, 1, 2, ..., n$.
A different approach of solving linear two-point boundary value problem has been suggested first, by Bickley in 1968 [4], he used cubic spline interpolation to model the solution curve and applied to the differential equations as well as the boundary conditions. Following this, Albasiny and Hoskins [1] applied the cubic spline interpolation which was introduced by Ahlberg et al. Fyfe [9] worked on this approach and concluded that spline method is better than the usual finite difference method. Caglar [6] proposed the use of cubic B-spline interpolation to solve this problem. Spline solution for regular boundary value problems have been used by many authors[11],[17], in [17] the non-polynomial cubic spline has been used to develop second and fourth order methods. Many authors used spline for numerical solution of singular two-point boundary value problems[14], the cubic spline has been used by [18],[19] and extended cubic B-spline used by [10] without any convergence analysis and comparison.
Recently the extended cubic B-spline methods have been used for solution boundary value problem in [15], but in this paper, cubic B-spline which contains a parameter λ has been used to solve boundary value problem, without any comparison and convergence analysis. Parametric cubic spline solution for linear second order boundary value problem has been used to develop fourth order method for a specific choice of the parameter by [3].
In this paper the derivation of quartic B-spline is presented in section 2. The numerical method based on quartic B-spline for solving two-point boundary value problem is given in section 3. Error analysis is presented in section 4. In section 5, convergence analysis of the presented method discussed which is based on Green's function approach and two-step method. In section 6, Numerical application of the method is illustrated by two test examples to demonstrate the efficiency of the method. Conclusion is given in section 7.

2. Quartic *B*-Spline

We consider the uniform grid partition $\Delta = \{x_0, x_1, \cdots, x_n\}$ of interval $[a, b]$, with mesh size $h = \frac{b-a}{n}$. Let s^4_Δ be the space of quartic B-spline with respect to Δ and with smoothness $C^3[a, b]$. The quartic B-splines are defined on $n+2$ nodes over the problem domain plus 8 additional nodes outside the problem domain $[a, b]$. These additional nodes are positioned as:

$$x_{-4} < x_{-3} < x_{-2} < x_{-1} < x_0 \text{ and } x_n < x_{n+1} < x_{n+2} < x_{n+3} < x_{n+4}$$

Donate the basis functions for quartic B-splines by $\phi_i(x)$, $i = -1, 0, 1..., n + 1, n + 2$.

$$\phi_i(x) = \frac{1}{h^4}\begin{cases} (x-x_{i-3})^4 & x_{i-3} \leqslant x \leqslant x_{i-2} \\ (x-x_{i-3})^4 - 5(x-x_{i-2})^4 & x_{i-2} \leqslant x \leqslant x_{i-1} \\ (x-x_{i-3})^4 - 5(x-x_{i-2})^4 + 10(x-x_{i-1})^4 & x_{i-1} \leqslant x \leqslant x_i \\ (x_{i+2}-x)^4 - 5(x_{i+1}-x)^4 & x_i \leqslant x \leqslant x_{i+1} \\ (x_{i+2}-x)^4 & x_{i+1} \leqslant x \leqslant x_{i+2} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

are quartic B-splines that vanish outside of the interval $[x_{i-3}, x_{i+2}]$ and is positive on the interior of that interval, that is $\phi_i(x) > 0$ for $x_{i-3} \leqslant x \leqslant x_{i+2}$ and provides a local partition of unity, that is $\sum_i \phi_i(x) = 1$ on Δ.

By using the above equation we have these properties too:

$$\phi_i(x_j) = \begin{cases} 11 \; i-j=0, i-j=1 \\ 1 \quad i-j=-1, i-j=2 \\ 0 \quad i-j=-2, \end{cases}$$

$$\phi_i'(x_j) = \begin{cases} \frac{12}{h}, \frac{-12}{h} & i-j=0, i-j=1 \\ \frac{4}{h}, \frac{-4}{h} & i-j=-1, i-j=2 \\ 0 & i-j=-2, \end{cases}$$

$$\phi_i''(x_j) = \begin{cases} \frac{-12}{h^2} & i-j=0, i-j=1 \\ \frac{12}{h^2} & i-j=-1, i-j=2 \\ 0 & i-j=-2. \end{cases}$$

3. Numerical Method for Boundary Value Problems

We consider a second-order two-point the following boundary value problem (1),(2). Based on collocation approach the solution of (1),(2) can be approximated by:

$$s(x) = \sum_{i=-1}^{n+2} c_i \phi_i(x), \tag{4}$$

where c_i are the unknown real coefficients and ϕ_i is quartic B-spline.

$$s(x_j) \approx y(x_j) = \sum_{i=-1}^{n+2} c_i \phi_i(x_j), \qquad 0 \leqslant j \leqslant n, \tag{5}$$

Thus the approximation at the point x_j can be written as

$$s(x_j) = c_{i-1}\phi_{i-1}(x_j) + c_i\phi_i(x_j) + c_{i+1}\phi_{i+1}(x_j) + c_{i+2}\phi_{i+2}(x_j) \approx y(x_j), \quad (6)$$

We can easily get

$$s'(x_j) = c_{i-1}\phi'_{i-1}(x_j) + c_i\phi'_i(x_j) + c_{i+1}\phi'_{i+1}(x_j) + c_{i+2}\phi'_{i+2}(x_j) \approx y'(x_j), \quad (7)$$

$$s''(x_j) = c_{i-1}\phi''_{i-1}(x_j) + c_i\phi''_i(x_j) + c_{i+1}\phi''_{i+1}(x_j) + c_{i+2}\phi''_{i+2}(x_j) \approx y''(x_j). \quad (8)$$

By substituting the relation (3) into eqs. (6)-(8) we have these relations:

$$\begin{aligned} s(x_i) &= c_{i-1} + 11c_i + 11c_{i+1} + c_{i+2}, \\ s'(x_i) &= \frac{4}{h}(-c_{i-1} - 3c_i + 3c_{i+1} + c_{i+2}), \\ s''(x_i) &= \frac{12}{h^2}(c_{i-1} - c_i - c_{i+1} + c_{i+2}). \end{aligned}$$

By substituting eqs. (6)-(8) into eqs. (1),(2) we obtain

$$s''(x_j) + p(x_j)s'(x_j) + q(x_j)s(x_j) = r(x_j), \qquad 0 \leqslant j \leqslant n, \quad (9)$$

$$s(x_0) = \alpha, s(x_n) = \beta,$$

By using equation (5) into equation (9) we have:

$$\sum_{i=-1}^{n+2} c_i[\phi''_i(x_j) + p(x_j)\phi'_i(x_j) + q(x_j)\phi_i(x_j)] = r(x_j), \qquad 0 \leqslant j \leqslant n, \quad (10)$$

solving the collocation equation (10) leads to the $(n+1)$ linear equations in $(n+4)$ unknowns. So we can obtain equation (11) for $0 \leqslant i \leqslant n$:

$$\begin{aligned} r_i = {} & \frac{1}{h^2}(12 - 4hp_i + q_ih^2)c_{i-1} + \frac{1}{h^2}(-12 - 12hp_i + 11q_ih^2)c_i \\ & + \frac{1}{h^2}(-12 + 12hp_i + 11q_ih^2)c_{i+1} + \frac{1}{h^2}(12 + 4hp_i + q_ih^2)c_{i+2}. \end{aligned} \quad (11)$$

By using the given boundary conditions yield to the following equations:

$$c_{-1} + 11c_0 + 11c_1 + c_2 = \alpha, \tag{12}$$

$$c_{n-1} + 11c_n + 11c_{n+1} + c_{n+2} = \beta. \tag{13}$$

The eqs. (12),(13) associated with equation (11) lead to the $(n+3)$ linear equations in $(n+4)$ unknowns, $C = (c_{-1}, c_0, ..., c_{n+2})^T$.
For solving this system we need to one equation, because of this we use the midpoints of subintervals. Consider the set of collocation points [12]:

$\Gamma = \{\tau_0 = x_0, \tau_1 = \frac{x_0+x_1}{2}, ..., \tau_i = \frac{x_{i-1}+x_i}{2}, ..., \tau_n = \frac{x_{n-1}+x_n}{2}, \tau_{n+1} = x_n\}$

That Γ includes the midpoints of subintervals and we have $x = x_0 + \frac{h}{2}$
By using relation (3) we have these properties:

$$\phi_i(x_j) = \begin{cases} \frac{76}{16} & i-j=0, i-j=-2 \\ \frac{1}{16} & i-j=1, i-j=-3 \\ \frac{230}{16} & i-j=-1, \end{cases}$$

$$\phi_i'(x_j) = \begin{cases} \frac{-11}{h}, \frac{11}{h} & i-j=0, i-j=-2 \\ \frac{-1}{2h}, \frac{1}{2h} & i-j=1, i-j=-3 \\ 0 & i-j=-1, \end{cases}$$

$$\phi_i''(x_j) = \begin{cases} \frac{12}{h^2} & i-j=0, i-j=-2 \\ \frac{3}{h^2} & i-j=1, i-j=-3 \\ \frac{-30}{h^2} & i-j=-1. \end{cases}$$

By substituting these properties into the boundary problems (1),(2) we obtain equation (14) for $\tau_1 = x_0 + \frac{h}{2}$.

$$\begin{aligned} r(x_0 + \frac{h}{2}) = & \frac{1}{h^2}(3c_{-1} + 12c_0 - 30c_1 + 12c_2 + 3c_3) \\ & + \frac{p(x_0 + \frac{h}{2})}{h}(\frac{-1}{2}c_{-1} - 11c_0 + 11c_2 + \frac{1}{2}c_3) \\ & + \frac{q(x_0 + \frac{h}{2})}{16}(c_{-1} + 76c_0 + 230c_1 + 76c_2 + c_3), \end{aligned} \tag{14}$$

Now by eliminating c_{-1}, c_{n+2} from eqs. (11)-(13) we obtain:

$$r_0 - \frac{\alpha}{h^2}(12 - 4hp_0 + q_0h^2) = \frac{1}{h^2}[(-144 + 32hp_0)c_0 + (-144 + 56hp_0)c_1 + (8hp_0)c_2], \tag{15}$$

and

$$r_n - \frac{\beta}{h^2}(12 + 4hp_n + q_nh^2) = \frac{1}{h^2}(-144 - 32hp_n)c_{n+1} + \frac{1}{h^2}(-144 - 56hp_n)c_n + \frac{1}{h^2}(-8hp_n)c_{n-1}, \tag{16}$$

and

$$r_i = \frac{1}{h^2}(12 - 4hp_i + q_ih^2)c_{i-1} + \frac{1}{h^2}(-12 - 12hp_i + 11q_ih^2)c_i + \frac{1}{h^2}(-12 + 12hp_i + 11q_ih^2)c_{i+1} + \frac{1}{h^2}(12 + 4hp_i + q_ih^2)c_{i+2}. \tag{17}$$

The equation (17) is for $1 \leqslant i \leqslant n-1$. Finally eqs.(14)-(16) associated by (17) lead to $(n+2) \times (n+2)$ linear system, this system can be solved by any Gausse eliminations or any iteration methods.

4. Error Analysis

Theorem 1: Let S be the quartic spline interpolating of $y \in C^{10}[a,b]$, defined by as follows

$$s_i = y_i, 1 \leqslant i \leqslant n, \tag{18}$$

$$s^{(4)}(x_i) = y^{(4)}(x_i) - \frac{h^2}{24}y^{(6)}(x_i) + \frac{7h^4}{5760}y^{(8)}(x_i), i = 1, 2, n-1, n. \tag{19}$$

So the following relations hold for $i = 1(1)n$.

$$s'(x_i) = y'(x_i) - \frac{7h^4}{5760}y^{(5)}(x_i) + O(h^6), \tag{20}$$

$$s''(x_i) = y''(x_i) + \frac{7h^4}{1920}y^{(6)}(x_i) + O(h^6). \tag{21}$$

Proof:
By substituting the relation (3) into eqs.(6)-(8) we have these relations:

$$\begin{aligned} s(x_i) &= c_{i-2} + 11c_{i-1} + 11c_i + c_{i+1}, \\ s'(x_i) &= \frac{4}{h}(-c_{i-2} - 3c_{i-1} + 3c_i + c_{i+1}), \\ s''(x_i) &= \frac{12}{h^2}(c_{i-2} - c_{i-1} - c_i + c_{i+1}). \end{aligned}$$

So these relations can be obtained.

$$s'_i = \frac{1}{h}(s_{i+1} - s_i) - \frac{h}{6}(s''_{i+1} + 2s''_i) + \frac{h^3}{384}(7s^{(4)}_{i+1} + 9s^{(4)}_i), \qquad 1 \leqslant i \leqslant n-1, \quad (22)$$

$$s''_i = \frac{1}{h^2}(s_{i-1} - 2s_i + s_{i+1}) - \frac{h^2}{384}(s^{(4)}_{i-1} + 30s^{(4)}_i + s^{(4)}_{i+1}), \qquad 2 \leqslant i \leqslant n-1. \quad (23)$$

To prove the relation (20), first we consider the relation (22) and by using the operator notation $E^0 s(x_i) = s(x_i)$, the relation (22) can be written in the following form:

$$s'(x_i) = \frac{1}{h}(E-1)y(x_i) - \frac{h}{6}(E+2)y''(x_i) + \frac{h^3}{384}(7E+9)y^{(4)}(x_i), \qquad (24)$$

By donating $D = \frac{d}{dx}$, the shift operator E can be expressed in term of D by $E = e^{hD}$ or

$$\begin{aligned} E = e^{hD} &= 1 + (hD) + \frac{(hD)^2}{2!} + \frac{(hD)^3}{3!} + \frac{(hD)^4}{4!} + \frac{(hD)^5}{5!} + \frac{(hD)^6}{6!} + ..., \\ E^{-1} = e^{-hD} &= 1 - (hD) + \frac{(hD)^2}{2!} - \frac{(hD)^3}{3!} + \frac{(hD)^4}{4!} - \frac{(hD)^5}{5!} + \frac{(hD)^6}{6!} + ..., \end{aligned}$$

Therefor, by using these expansions , the equation (24) can be simplified into

$$\begin{aligned} s'(x_i) = &\frac{1}{h}((hD) + \frac{(hD)^2}{2!} + \frac{(hD)^3}{3!} + \frac{(hD)^4}{4!} + \frac{(hD)^5}{5!} + \frac{(hD)^6}{6!} + ...)y(x_i) \\ &- \frac{h}{6}(3 + (hD) + \frac{(hD)^2}{2!} + \frac{(hD)^3}{3!} + \frac{(hD)^4}{4!} + \frac{(hD)^5}{5!} + \frac{(hD)^6}{6!} + ...)y''(x_i) \\ &+ \frac{h^3}{384}(16 + 7(hD) + 7\frac{(hD)^2}{2!} + 7\frac{(hD)^3}{3!} + 7\frac{(hD)^4}{4!} + 7\frac{(hD)^5}{5!} + 7\frac{(hD)^6}{6!} + ...)y^{(4)}(x_i), \end{aligned}$$

Hence we obtained the relation (20) as

$$s'(x_i) = y'(x_i) - \frac{7h^4}{5760}y^{(5)}(x_i) + O(h^6).$$

Now we have to prove the relation (21), by using the relations (19),(23) we obtain

$$s''_i = \frac{1}{h^2}(s_{i-1} - 2s_i + s_{i+1}) - \frac{h^2}{384}[(y^{(4)}_{i-1} - \frac{h^2}{24}y^{(6)}_{i-1} + \frac{7h^4}{5760}y^{(8)}_{i-1} + O(h^6))$$
$$+ 30(y^{(4)}_i - \frac{h^2}{24}y^{(6)}_i + \frac{7h^4}{5760}y^{(8)}_i + O(h^6)) + (y^{(4)}_{i+1} - \frac{h^2}{24}y^{(6)}_{i+1} + \frac{7h^4}{5760}y^{(8)}_{i+1} + O(h^6))],$$

So by same approach we have:

$$s''_i = \frac{1}{h^2}(E^{-1} + E - 2)y(x_i) - \frac{h^2}{384}(E^{-1}y^{(4)}_i - \frac{h^2}{24}E^{-1}y^{(6)}_i + \frac{7h^4}{5760}E^{-1}y^{(8)}_i + O(h^6))$$
$$- \frac{30h^2}{384}(y^{(4)}_i - \frac{h^2}{24}y^{(6)}_i + \frac{7h^4}{5760}y^{(8)}_i + O(h^6))$$
$$- \frac{h^2}{384}(Ey^{(4)}_i - \frac{h^2}{24}Ey^{(6)}_i + \frac{7h^4}{5760}Ey^{(8)}_i + O(h^6)),$$

and

$$s''_i = \frac{1}{h^2}((hD)^2 + 2\frac{(hD)^4}{4!} + 2\frac{(hD)^6}{6!} + ...)y(x_i)$$
$$- \frac{h^2}{384}(32 + (hD)^2 + 2\frac{(hD)^4}{4!} + 2\frac{(hD)^6}{6!} + ...)y^{(4)}(x_i)$$
$$- \frac{h^4}{9216}(32 + (hD)^2 + 2\frac{(hD)^4}{4!} + 2\frac{(hD)^6}{6!} + ...)y^{(6)}(x_i)$$
$$- \frac{7h^6}{2211840}(32 + (hD)^2 + 2\frac{(hD)^4}{4!} + 2\frac{(hD)^6}{6!} + ...)y^{(8)}(x_i) + O(h^6),$$

Hence

$$s''(x_i) = y''(x_i) + \frac{7h^4}{1920}y^{(6)}(x_i) + O(h^6).$$

The proof of relation (21) is completed.

Theorem 2: Let S be the quartic spline interpolating of $y \in C^{10}[a, b]$, defined by eqs. (18),(19) the following relations hold on the grid points $x_i, i = 0(1)n$.

$$s(x_i) = y(x_i) + O(h^6), \tag{25}$$

$$s'(x_i) = y'(x_i) + \frac{h^4}{720}y^{(5)}(x_i) + O(h^6), \tag{26}$$

$$s''(x_i) = y''(x_i) - \frac{h^4}{240} y^{(6)}(x_i) + O(h^6). \tag{27}$$

Proof:
The proof of this theorem is similar to theorem 1. But the following remark can be obtained from this theorem.
Further we define two discrete difference operators δg_i and $\delta^2 g_i$ which will be used to formulate the quartic spline collocation methods:

$$\delta g_i = g_{i-1} - 2g_i + g_{i+1}, 2 \leqslant i \leqslant n-1,$$
$$\delta^2 g_i = g_{i-2} - 4g_{i-1} + 6g_i - 4g_{i+1} + g_{i+2}, 3 \leqslant i \leqslant n-2.$$

From eqs. (20),(21) and (25)-(27) and by [21] the following relations can be obtained

$$y_i^{(5)} = \frac{1}{h^4} \delta^2 s_i' + O(h^2), 3 \leqslant i \leqslant n-2, \tag{28}$$

$$y_i^{(6)} = \frac{1}{h^4} \delta^2 s_i'' + O(h^2), 3 \leqslant i \leqslant n-2. \tag{29}$$

If $y \in C^{10}[a,b]$ then the following approximations to $y^{(5)}$ and $y^{(6)}$ are obtained at boundary and near-boundary points $\{x_0, \tau_1, \tau_2, \tau_{n-1}, \tau_n, x_0\}$ for $k = 5, 6$:

$$y^{(k)}(x_0) = \frac{1}{2h^4}(7\delta^2 s_3^{(k-4)} - 5\delta^2 s_4^{(k-4)}) + O(h^2),$$
$$y^{(k)}(\tau_1) = \frac{1}{h^4}(3\delta^2 s_3^{(k-4)} - 2\delta^2 s_4^{(k-4)}) + O(h^2),$$
$$y^{(k)}(\tau_2) = \frac{1}{h^4}(2\delta^2 s_3^{(k-4)} - \delta^2 s_4^{(k-4)}) + O(h^2),$$
$$y^{(k)}(\tau_{n-1}) = \frac{1}{h^4}(2\delta^2 s_{n-2}^{(k-4)} - \delta^2 s_{n-3}^{(k-4)}) + O(h^2),$$
$$y^{(k)}(\tau_n) = \frac{1}{h^4}(3\delta^2 s_{n-2}^{(k-4)} - 2\delta^2 s_{n-3}^{(k-4)}) + O(h^2),$$
$$y^{(k)}(x_n) = \frac{1}{2h^4}(7\delta^2 s_{n-2}^{(k-4)} - 5\delta^2 s_{n-3}^{(k-4)}) + O(h^2),$$
$$y^{(k)}(x_1) = \frac{1}{2h^4}(5\delta^2 s_3^{(k-4)} - 3\delta^2 s_4^{(k-4)}) + O(h^2),$$
$$y^{(k)}(x_{n-1}) = \frac{1}{2h^4}(5\delta^2 s_{n-2}^{(k-4)} - 3\delta^2 s_{n-3}^{(k-4)}) + O(h^2).$$

5. Convergence Analysis of the Quartic Spline Collocation Method

There are two optimal quartic spline collocation methods, the one-step and the two-step collocation methods. Here we describe the two-step collocation method. The two-step collocation method is defined by the following steps:

Step 1:
Determine $\nu \subset s^4_\Delta$ such that it satisfies

$$[L\nu - r]_{\tau_i} = 0, 1 \leqslant i \leqslant n, \tag{30}$$

$$[B\nu - g]_{x_0, x_n} = 0, \tag{31}$$

$$[L\nu - r]_{x_1, x_{n-1}} = 0. \tag{32}$$

Step 2:
Determine $u_\Delta \subset s^4_\Delta$ such that it satisfies

$$[Lu_\Delta - \bar{r}]_{\tau_i} = 0, 1 \leqslant i \leqslant n, \tag{33}$$

$$[Bu_\Delta - \bar{g}]_{x_0, x_n} = 0, \tag{34}$$

$$[Lu_\Delta - \bar{r}]_{x_1, x_{n-1}} = 0. \tag{35}$$

where

$$\bar{r}_i = r_i + \frac{7}{1920}\delta^2\nu_i^{(2)} - \frac{7}{5760}p_i\delta^2\nu_i^{(1)}, 1 \leqslant i \leqslant n,$$
$$\bar{r}_1 = r_1 + \frac{7}{1920}(3\delta^2\nu_3^{(2)} - 2\delta^2\nu_4^{(2)}) - \frac{7}{5760}p_1(3\delta^2\nu_3^{(1)} - 2\delta^2\nu_4^{(1)}),$$
$$\bar{r}_2 = r_2 + \frac{7}{1920}(2\delta^2\nu_3^{(2)} - \delta^2\nu_4^{(2)}) - \frac{7}{5760}p_2(2\delta^2\nu_3^{(1)} - \delta^2\nu_4^{(1)}),$$
$$\bar{r}_{n-1} = r_{n-1} + \frac{7}{1920}(2\delta^2\nu_{n-2}^{(2)} - \delta^2\nu_{n-3}^{(2)}) - \frac{7}{5760}p_{n-1}(2\delta^2\nu_{n-2}^{(1)} - \delta^2\nu_{n-3}^{(1)}),$$
$$\bar{r}_n = r_n + \frac{7}{1920}(3\delta^2\nu_{n-2}^{(2)} - 2\delta^2\nu_{n-3}^{(2)}) - \frac{7}{5760}p_n(3\delta^2\nu_{n-2}^{(1)} - 2\delta^2\nu_{n-3}^{(1)}),$$
$$\bar{r}(x_1) = r(x_1) - \frac{1}{480}(5\delta^2\nu_3^{(2)} - 3\delta^2\nu_4^{(2)}) + \frac{1}{1440}p(x_1)(5\delta^2\nu_3^{(1)} - 3\delta^2\nu_4^{(1)}),$$
$$\bar{r}(x_{n-1}) = r(x_{n-1}) - \frac{1}{480}(5\delta^2 s_{n-2}^{(2)} - 3\delta^2 s_{n-3}^{(2)}) + \frac{1}{1440}p(x_{n-1})(5\delta^2\nu_{n-2}^{(1)} - 3\delta^2\nu_{n-3}^{(1)}),$$
$$\bar{g}(x_0) = g(x_0) = \alpha,$$
$$\bar{g}(x_n) = g(x_n) = \beta.$$

Green's function approach:
We will proceed to the convergence analysis of the purposed method via Green's function. If we assume that the boundary value problem $y'' = 0$, $By = 0$ has a unique solution, then it implies that there is a Green's function $G(x,t)$ for this problem [20]. Let $y'' = \varphi$ and $\hat{s}'' = \psi$ are the exact and spline solutions of the given problem which satisfy the boundary conditions, we assume $\nu'' = \eta$. Then $y(x)$ and $\hat{s}(x)$ can be obtained in the following forms:

$$y(x) = \int_a^b G(x,t)\varphi(t)dt,$$
$$y'(x) = \int_a^b G_x(x,t)\varphi(t)dt,$$
$$\hat{s}(x) = \int_a^b G(x,t)\psi(t)dt,$$
$$\hat{s}'(x) = \int_a^b G_x(x,t)\psi(t)dt,$$
$$\nu(x) = \int_a^b G(x,t)\eta(t)dt,$$
$$\nu'(x) = \int_a^b G_x(x,t)\eta(t)dt.$$

We introduce the operator k, $k : C[a,b] \to C[a,b]$ that is defined by

$$k\varphi(x) = p(x)\int_a^b G_x(x,t)\varphi(t)dt + q(x)\int_a^b G(x,t)\varphi(t)dt.$$

We also introduce the linear projection p_Δ that maps $C[a,b]$ to s''_Δ by piecewise quartic interpolation at the midpoints $\{\tau_i\}_1^n$ and grid points x_0, x_n, i.e , at points $\{\tau_i\}_0^{n+1}$ since $\tau_0 = x_0$ and $\tau_{n+1} = x_n$.

Convergence analysis of the two-step method:
We present the convergence analysis and error bounds for the two step method by using a Green's function approach. With the notations introduced, we can rewrite eqs. (30)-(32) and (33)-(35) respectively as

$$p_\Delta(\eta + k\eta) = p_\Delta r, \tag{36}$$

$$p_\Delta(\psi + k\psi) = p_\Delta \bar{r}. \tag{37}$$

Since $p_\Delta\eta = \eta$ and $p_\Delta\psi = \psi$, we can simplify eqs. (36),(37) as

$$(I + p_\Delta)\eta = p_\Delta r, \tag{38}$$

$$(I + p_\Delta)\psi = p_\Delta \bar{r}. \tag{39}$$

The equation (1) can be rewritten as

$$\varphi + k\varphi = r.$$

By the definition of p_Δ, $\|p_\Delta\varphi - \varphi\|_\infty$ converges to zero as h approaches zero for continuous function φ. By the complete continuity of k, this implies that $\|p_\Delta k - k\|_\infty$ converges to zero as h converges to zero. Therefore by Neumann's theorem, we conclude that the operators $(I + p_\Delta k)^{-1}$ exist and are uniformly bounded for sufficient small h.

Theorem 3: we assume the functions $p(x), q(x)$ and $r(x)$ are given in equation (1),(2) has a unique solution in $C^4[a, b]$, also the test problem $y'' = 0$ with boundaries vanish at the a and b (that is $By = 0$) has a unique solution, then the collocation approximation $u_\Delta \in s_\Delta^{(4)}$ defined in eqs. (33)-(35) exists and the global error satisfies:

$$\|(y - u_\Delta)^{(k)}\|_\infty = O(h^{5-k}), k = 0, 1, 2.$$

Proof:
From the existence and uniformly bounded of $(I + p_\Delta k)^{-1}$, the solvability of the relations (38),(39) follows, hence the unique existence of u_Δ follows.
Recall the quartic spline interplant S of y in eqs. (18),(19). If $\nu \in s_\Delta^{(4)}$ defined in eqs. (30)-(32) so we have the relation for $k = 0, 1, 2$ and $2 \leqslant i \leqslant n - 1$

$$\delta^2 S_i^{(k)} = \delta^2 \nu_i^{(k)} + O(h^6)$$

Therefore,for u_Δ defined in eqs. (33)-(35) we have:

$$L(S - u_\Delta)(\tau_i) = O(h^6), 1 \leqslant i \leqslant n, \tag{40}$$

$$L(S - u_\Delta)(x_i) = O(h^6), i = 0, n, \tag{41}$$

$$B(S - u_\Delta) = O(h^6). \tag{42}$$

Note that there exists a linear function w such that $Bw = B(S - u_\Delta) = O(h^6)$ because of assumption. It can be further shown that $\|w\|_\infty = O(h^6)$ and $\|w'\|_\infty = O(h^6)$. Then we can rewrite eqs. (40)-(42) as

$$(I + p_\Delta k)(S'' - w'' - u''_\Delta) = O(h^6),$$

From the uniformly bounded of $(I + p_\Delta k)^{-1}$, we obtain

$$\|S'' - w'' - u''_\Delta\|_\infty = O(h^6). \tag{43}$$

Since the unique solvability of $(S-w-u_\Delta)'' = (S-u_\Delta)'' = 0$, $B(S-w-u_\Delta) = 0$ is ensured by assumption, we can obtain by using the Green's function:

$$(S - w - u_\Delta)'(x) = \int_a^b G_x(x,t)(S'' - w'' - u''_\Delta)(t)dt,$$
$$(S - w - u_\Delta)(x) = \int_a^b G(x,t)(S'' - w'' - u''_\Delta)(t)dt.$$

These imply that

$$\|S - w - u_\Delta\|_\infty = O(h^6), \tag{44}$$

$$\|S' - w' - u'_\Delta\|_\infty = O(h^6), \tag{45}$$

From eqs. (43)-(45) and the definition of w, we utilize the triangle inequality to obtain

$$\|S^{(k)} - u_\Delta^{(k)}\|_\infty \leqslant \|S^{(k)} - w^{(k)} - u_\Delta^{(k)}\|_\infty + \|w^{(k)}\|_\infty = O(h^6).$$

This completes the proof.

6. Numerical Illustrations

we consider two examples of second order two-point boundary value problems from [8].

Example 1

$$y''(x) - y'(x) = -e^{(x-1)} - 1,$$

$$y(0) = y(1) = 0,$$

With the exact solution $y(x) = x(1 - e^{(x-1)})$.

Example 2

$$y''(x) - y'(x) = e^{(1-x)}(x^{\frac{3}{2}}(1-x)^{\frac{3}{2}}(60-20x) - 15x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}(1-2x+2x^2)),$$

$$y(0) = y(1) = 0,$$

With the exact solution $y(x) = 4x^{\frac{5}{2}}(1-x)^{\frac{5}{2}}$.
To verify the applicability of our purposed method and to show the accurate nature of our approach. These two test problems have been solved with different steps size $h = 0.1, 0.01$. The computed results are compared with exact results and absolute errors in the solution are tabulated in tables $1, 2$.
we compared our results of example (1) with finite difference solutions in [8], B-spline interpolation in [6] and Cubic spline method in [17], in order to we compared our results of example (2) with finite difference solutions in [8] and Cubic spline method in [17].

Table 1. Comparison of the maximum absolute error in the solution of Example1

h	our method	finite difference[8]	B-spline interpolation[6]	Cubic spline[17]
0.1	2.6672×10^{-7}	$8.24e-3$	$2.9e-4$	$1.88e-3$
0.01	2.67275×10^{-11}	$8.31e-3$	$2.89e-6$	$1.87e-4$

Table 2. Comparison of the maximum absolute error in the solution of Example2

h	our method	finite difference[8]	Cubic spline[17]
0.1	1.46214×10^{-3}	$3.50e-1$	$2.81e-1$
0.01	3.90394×10^{-6}	$2.45e-1$	$1.25e-1$

7. Conclusion

In this work we formulate quartic B-spline for collocation of two-point boundary value problem. Convergence analysis of the presented method is discussed, the method applied to two test examples, the absolute errors in the solution obtained by our method are compared with the method in [2],[10],[11]. We find that our results are considerable accurate.

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