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Generalization of Titchmarsh's Theorem for the Dunkl Transform in the Space $L^p_\alpha(\mathbb{R})$

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Abstract. In this paper, using a generalized Dunkl translation operator, we obtain a generalization of Titchmarsh's Theorem for the Dunkl transform for functions satisfying the(ψ, p)-Lipschitz Dunkl condition in the space $L_{p,\alpha} = L^p(\mathbb{R}, |x|^{2\alpha+1}dx)$, where $\alpha > -\frac{1}{2}$.

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1. Introduction and preliminaries

Dunkl operators are differential-difference operators introduced in 1989, by Dunkl [2]. On the real line, these operators, which are denoted by D_α, depend on a real parameter $\alpha > -\frac{1}{2}$.
In [1], we proved an analog of Titchmarsh's theorem for the Dunkl transform in the space $L_{2,\alpha}$. In this paper we prove a generalization of this theorem in the space $L_{p,\alpha}$, where $1 < p \leqslant 2$. For this purpose, we use a generalized Dunkl translation operator.

$L_{p,\alpha} = L^p(\mathbb{R}, |x|^{2\alpha+1}dx)$; $1 < p \leqslant 2$, is the Banach space of measurable functions $f(t)$ on $\mathbb{R}$ with finite norm

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$$\|f\|_{p,\alpha} = \left(\int_{-\infty}^{\infty} |f(x)|^p |x|^{2\alpha+1} dx\right)^{1/p}.$$

The Dunkl operator is a differential-difference operator D_α

$$\mathrm{D}_\alpha = \frac{df(x)}{dx} + (\alpha + \frac{1}{2})\frac{f(x) - f(-x)}{x},\ \alpha > -\frac{1}{2},$$

where $f \in \mathrm{L}_{p,\alpha}$.

Let $j_\alpha(t)$ is a normalized Bessel function of the first kind

$$j_\alpha(t) = \frac{2^\alpha \Gamma(\alpha+1) J_\alpha(t)}{t^\alpha},$$

where $J_\alpha(t)$ is a Bessel function of the first kind. The function $j_\alpha(t)$ is infinitely differentiable and even.

The Dunkl kernel defined by

$$e_\alpha(x) = j_\alpha(x) + ic_\alpha j_{\alpha+1}(x),$$

where $c_\alpha = (2\alpha+2)^{-1}$.

Using the correlation

$$j'_\alpha(x) = -\frac{x j_{\alpha+1}(x)}{2(\alpha+1)}.$$

We have

$$e_\alpha(x) = j_\alpha(x) - ij'_\alpha(x). \tag{1}$$

The Dunkl transform is defined by

$$\widehat{f}(\lambda) = \int_{-\infty}^{\infty} f(x) e_\alpha(\lambda x) |x|^{2\alpha+1} dx,\ \lambda \in \mathbb{R}.$$

The inverse Dunkl transform is defined by the formula

$$f(x) = (2^{\alpha+1}\Gamma(\alpha+1))^{-2} \int_{-\infty}^{\infty} \widehat{f}(\lambda) e_\alpha(-\lambda x) |\lambda|^{2\alpha+1} d\lambda.$$

Plancherel's theorem and the Marcinkiewics interpolation theorem (see [3]) we get for $f \in \mathrm{L}_{p,\alpha}$ with $1 < p \leqslant 2$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|\widehat{f}\|_{q,\alpha} \leqslant C\|f\|_{p,\alpha}, \tag{2}$$

where C is a positive constant.

K. Trimèche has introduced in [4] the generalized Dunkl translation operator τ_h, $h \in \mathbb{R}$, we have

$$\widehat{(\tau_h f)}(x) = e_\alpha(xh)\widehat{f}(x). \tag{3}$$

The function $j_\alpha(x)$ is defined also by

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{2n}}{n!\Gamma(n+\alpha+1)},\ z \in \mathbb{C}. \tag{4}$$

Moreover, from (4) we see that

$$\lim_{z \longrightarrow 0}\frac{j_\alpha(z)-1}{z^2} \neq 0$$

by consequence, there exist $c > 0$ and $\eta > 0$ satisfying

$$|z| \leqslant \eta \Longrightarrow |j_\alpha(z) - 1| \geqslant c|z|^2. \tag{5}$$

2. Main Result

In this section we give the main result of this paper. We need first to define (ψ, p)-Lipschitz Dunkl class.

Definition 2.1 *A function $f \in \mathrm{L}_{p,\alpha}$ is said to be in the (ψ, p)-Lipschitz Dunkl class, denoted by $Lip(\psi, p)$, if*

$$\|\tau_h f(x) + \tau_{-h} f(x) - 2f(x)\|_{p,\alpha} = O(\psi(h)) \text{ as } h \longrightarrow 0,$$

where $\psi(t)$ is a continuous increasing function on $[0,\infty)$, $\psi(0) = 0$ and $\psi(ts) = \psi(t)\psi(s)$ for all $t, s \in [0,\infty)$.

Theorem 2.2 *Let $f(x)$ belong to $Lip(\psi, p)$. Then*

$$\int_{|\lambda|\geqslant r} |\widehat{f}(\lambda)|^q|\lambda|^{2\alpha+1}d\lambda = O(\psi(r^{-q})) \text{ as } r \longrightarrow +\infty.$$

Proof Let $f \in Lip(\psi, p)$. Then we have

$$\|\tau_h f(x) + \tau_{-h} f(x) - 2f(x)\|_{p,\alpha} = O(\psi(h)) \text{ as } h \longrightarrow 0$$

From formulas (1) and (2), we have the Dunkl transform of $\tau_h f(x) + \tau_{-h} f(x) - 2f(x)$ is $2(j_\alpha(\lambda h) - 1)\widehat{f}(\lambda)$.

By (2), we obtain

$$\left(\int_{-\infty}^{\infty} 2^q |j_\alpha(\lambda h) - 1|^q |\widehat{f}(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda\right)^{1/q} \leqslant C\|\tau_h f(x) + \tau_{-h} f(x) - 2f(x)\|_{p,\alpha}$$

From (5), we have

$$\int_{\frac{\eta}{2h} \leqslant |\lambda| \leqslant \frac{\eta}{h}} |1 - j_\alpha(\lambda h)|^q |\widehat{f}(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda \geqslant \frac{c^q \eta^{2q}}{2^{2q}} \int_{\frac{\eta}{2h} \leqslant |\lambda| \leqslant \frac{\eta}{h}} |\widehat{f}(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda$$

There exists then a positives constants C_1 and K_1 such that

$$\begin{aligned}\int_{\frac{\eta}{2h} \leqslant |\lambda| \leqslant \frac{\eta}{h}} |\widehat{f}(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda &\leqslant C_1 \int_{-\infty}^{\infty} |1 - j_\alpha(\lambda h)|^q |\widehat{f}(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda \\ &\leqslant K_1 \psi^q(h) = K_1 \psi(h^q).\end{aligned}$$

Then

$$\int_{r \leqslant |\lambda| \leqslant 2r} |\widehat{f}(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda \leqslant K\psi(r^{-q}),$$

where $K = K_1 \psi(\eta^q 2^{-q})$.

Of course

$$\begin{aligned}\int_{|\lambda| \geqslant r} |\widehat{f}(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda &= \left(\int_{r \leqslant |\lambda| \leqslant 2r} + \int_{2r \leqslant |\lambda| \leqslant 4r} + \int_{4r \leqslant |\lambda| \leqslant 8r} + ...\right) |\widehat{f}(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda \\ &\leqslant K\psi(r^{-q}) + K\psi((2r)^{-q}) + K\psi((4r)^{-q}) + ... \\ &\leqslant K\psi(r^{-q}) + K\psi(2^{-q})\psi(r^{-q}) + K\psi((2^{-q})^2)\psi(r^{-q}) + ... \\ &\leqslant K\psi(r^{-q})(1 + \psi(2^{-q}) + \psi((2^{-q})^2) + ...).\end{aligned}$$

We have $\psi(2^{-q}) < 1$, then

$$\int_{|\lambda| \geqslant r} |\widehat{f}(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda \leqslant C_2 \psi(r^{-q}),$$

where $C_2 = K(1 - \psi(2^{-q}))^{-1}$.

Finally, we get

$$\int_{|\lambda|\geqslant r} |\widehat{f}(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda = O(\psi(r^{-q}))\ as\ r \longrightarrow \infty.$$

Thus, the proof is finished. ■

3. Conclusion

In this work we have succeded to generalise the theorem in [1] for the Dunkl transform in the space $\mathrm{L}_{p,\alpha}$. We proved that $f(x)$ belong to $Lip(\psi, p)$. Then

$$\int_{|\lambda|\geqslant r} |\widehat{f}(\lambda)|^q |\lambda|^{2\alpha+1} d\lambda = O(\psi(r^{-q}))\ as\ r \longrightarrow +\infty.$$

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