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A Modified Method to Determine a Well-Dispersed Subset of Non-Dominated Vectors of an MOMILP Problem

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Abstract. This paper uses the L_1-norm and the concept of the non-dominated vector, to propose a method to find a well-dispersed subset of non-dominated (WDSND) vectors of a multi-objective mixed integer linear programming (MOMILP) problem.

The proposed method generalizes the proposed approach by Tohidi and Razavyan [Tohidi G., S. Razavyan (2014), determining a well-dispersed subset of non-dominated vectors of multi-objective integer linear programming problem, International Journal of Industrial Mathematics, (Accepted for publication)] to find a WDSND vectors of an MOMILP problem.

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1. Introduction

Pasternak and Passy [4] conducted an earlier study on designing solving methods for Multi-objective integer linear programming (MOILP) problem. They used the concept of implicit enumeration to resolve 0-1 bi-criterion linear programs. Bitran [1] used relaxation techniques to generate non-dominated vectors. Bitran [1] also reported some computational results. Deckro and Winkfsky [2] reported computational results in terms of implicit enumeration compared to Bitran's works.

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Sylva and Crema [6] propose a method for finding a well-dispersed subset of non-dominated solutions based on maximizing the infinity norm distance from a set of known solutions. They claim that their approach originally provides a variant of the procedure by Sylva and Crema [5]. The major drawback of this approach is the difficulty of solving the constrained problems due to increasing number of constraints and binary variables. To reduce the computational efforts Tohidi and Razavyan [7] proposed a method to find a WDSND vectors of a MOILP problem using the concept of ideal point and L_1-norm.

This paper generalizes the proposed method by Tohidi and Razavyan [7] and develops a one-stage algorithm to find a WDSND vectors of an MOMILP problem. The proposed method determines at least one non-dominated vector in each iteration and it solves only a single objective mixed integer programming linear program in each iteration. Its iterations do not increase the number of constraints and variables of these single objective problems. The proposed algorithm reduces computational efforts for solving MOILP problem. For large problems, the improvement can be much more significant.

The organization of this paper is as follows. Section 2 presents a background MOMILP problem. Section 3 introduces the modified method for finding a well-dispersed subset of the non-dominated vectors of an MOMILP problem. Section 4 illustrates the procedure using a numerical example. Finally, conclusions are presented in Section 5.

2. Background

A Multi-Objective Programming Problem with s-Objectives is Defined as:$\Im$

$$\begin{array}{ll} \max & \big(f_1(W),\ldots,f_s(W)\big) \\ \text{s.t.} & W \in X \end{array} \tag{2.1}$$

where $f_1,\ldots,f_k$ are the objective functions and X is a feasible region.

Definition 1: $\overline{W} \in X$ is said to be an non-dominated vector of problem (2.1) if and only if there does not exist a point $W^o \in X$, such that:

$$\big(f_1(W^0),\ldots,f_s(W^0)\big) \geqslant \big(f_1(\overline{W}),\ldots,f_s(\overline{W})\big)$$

and the inequality holds strictly for at least one index.

When all constraints and objective functions are linear and some variables are integer, model (2.1) is called the MOMILP problem and is as follows:

$$\begin{array}{lll} \max & \{C_1W,\cdots,C_sW\} & \\ \text{s.t.} & A_iW \leqslant b_i, & i=1,2,\ldots,m \\ & W \geqslant 0, w_j \in Z, j \in J & \end{array} \tag{2.2}$$

where, $C_r = (c_{1r},\ldots,c_{nr})$ $(r=1,\ldots,s)$, $A_i = (a_{i1},\ldots,a_{in})$ $(i=1,\ldots,m)$, $J \subseteq \{1,\ldots,n\}$, $Z=\{\ldots,-2,-1,0,1,2,\ldots\}$ and $W=(w_1,\ldots,w_n)^T$. The set X, which is defined as follows:

$$X = \Big\{W \ \mid \ A_iW \leqslant b_i, W \in Z^n_+, w_j \in Z, j \in J, i=1,\ldots,m\Big\},$$

is called the set of feasible solutions of problem (2.2). Let X be bounded. Corresponding to each $W \in X$ the vector Y is defined as follows [3]:

$$Y = (y_1, \dots, y_s)^T = (C_1 W, \dots, C_s W)^T.$$

Definition 2: It is said that the vector $Y = (y_1 \dots, y_s)^T$ dominates the vector $Y^o = (y_1^o, \dots, y_s^o)^T$ if for each $r (r = 1, \dots, s)$, $y_r \geqslant y_r^o$ and there is at least one l such that $y_l > y_l^o$.

Definition 3: The set F, which is defined as

$$F = \left\{Y \mid Y = (C_1 W, \dots, C_s W)^T,\ A_i W \leqslant b_i,\ i = 1, \dots, m, W \geqslant 0, w_j \in Z, j \in J\right\},$$

is called the values space of the objective functions in problem (2.2).

Let $g_r = C_r W_r^* (r = 1, \dots, s)$ where W_r^* is the optimal solution of the $r^{th} (r = 1, \dots, s)$ problem from the following problems:

$$\begin{aligned} g_r = \max\ & C_r W \\ \text{s.t. } & A_i W \leqslant b_i, \qquad i = 1, \dots, m \\ & W \geqslant 0, w_j \in Z, j \in J. \end{aligned} \tag{2.3}$$

Definition 4: The vector g, which is defined as

$$g = (g_1, \dots, g_s)^T = (C_1 W_1^*, \dots, C_s W_s^*)^T,$$

is called the ideal vector [3].

3. Non-dominated Vectors of an MOMILP Problem

To find the non-dominated vectors of problem (2.2), we can specify $W \in X$ such that $g - Y = (g_1 - C_1 W, \dots, g_s - C_s W)^T$ is minimized [3]. Hence, the following problem is solved:

$$\begin{aligned} \min\ & \{g_1 - C_1 W, g_2 - C_2 W, \dots, g_s - C_s W\} \\ \text{s.t. } & A_i W \leqslant b_i, \quad i = 1, \dots, m \\ & W \geqslant 0, w_j \in Z, j \in J. \end{aligned} \tag{3.1}$$

For each $W \in X$, $g_r \geqslant C_r W$ $(r = 1, \dots, s)$, hence:

$$\min_{W \in X} \sum_{r=1}^{s} |g_r - C_r W| = \min_{W \in X} \sum_{r=1}^{s} (g_r - C_r W) = \sum_{r=1}^{s} g_r - \max_{W \in X} \sum_{r=1}^{s} C_r W.$$

Therefore, problem (3.1) is converted to the following mixed integer linear programming problem:

$$\begin{aligned} \theta_0^* = \max\ & \sum_{r=1}^{s} C_r W \\ \text{s.t. } & A_i W \leqslant b_i, \qquad i = 1, \dots, m \\ & W \geqslant 0, w_j \in Z, j \in J. \end{aligned} \tag{3.2}$$

Theorem 1: *Each optimal solution of problem (3.2) is a non-dominated vector for problem (2.2).*

Proof: The proof is similar to that of Theorem 1.3 in [3] and is not repeated here. □

Let $E_0 = \{W_1^*, \ldots, W_k^*\}$ be the set of the optimal solutions of problem (3.2) and $I_0 = \{1, \ldots, k\}$. If $X \neq \phi$, then $E_0 \neq \phi$. Suppose that $Y_j = (y_1^1, \ldots, y_1^s) = (C_1 W_j^*, \ldots, C_s W_j^*)^T, j \in I_0$. We consider the following set [7]:

$$\overline{Y}_0 = \Big\{ y | y \leqslant \sum_{j \in I_0} \lambda_j y_j, \sum_{j \in I_0} \lambda_j = 1, \lambda_j \in \{0,1\}, j \in I_0 \Big\}.$$

For $\overline{X} = X - E_0$ we have [7]:

$$\min_{W \in \overline{X}} \sum_{r=1}^{s} |g_r - C_r W| > \sum_{r=1}^{s} g_r + \theta_0^* \Longrightarrow \min_{W \in \overline{X}} \sum_{r=1}^{s} (-C_r W) > \theta_0^*.$$

Therefore, we have the following Theorem.

Theorem 2: *There is no non-dominated vector for (2.2), say $\widehat{W} \in \overline{X}$, with $\sum_{r=1}^{s} C_r \widehat{W} \geqslant \theta_0^*$.*

To find another non-dominated vectors of problem (2.2), we determine a non-dominated vector of problem (2.2), say $W_{k+1}^* \in X$, such that W_{k+1}^* is an optimal solution of the model $\min_{W \in X} \sum_{r=1}^{s} |g_r - C_r W|$. Therefore, $Y_{k+1} = (y_{k+1}^1, \ldots, y_{k+1}^s) = (C_1 W_{k+1}^*, \ldots, C_s W_{k+1}^*)^T \notin \overline{Y}_0$. That is the following inequalities are not satisfied simultaneously [7].

$$\begin{aligned} & y_{k+1}^1 \leqslant \sum_{j \in I_0} \lambda_j y_j^1 \\ & y_{k+1}^2 \leqslant \sum_{j \in I_0} \lambda_j y_j^2 \\ & \vdots \\ & y_{k+1}^s \leqslant \sum_{j \in I_0} \lambda_j y_j^s \end{aligned}$$

where $\sum_{j \in I_0} \lambda_j = 1$ and $\lambda_j \in \{0,1\}$. In other words $\exists i \in \{1, \ldots, s\}$ such that $y_{k+1}^i > \max_{j \in I_0} \{y_j^i\}$ [7].

Let M be a large positive real number and $\delta_i \in \{0,1\}$ for $i \in \{1, \ldots, s\}$. We can consider the following constraints which are satisfied simultaneously.

$$\begin{aligned} & y_{k+1}^1 > \max_{j \in I_0} \{y_j^1\} - M\delta_1 \\ & y_{k+1}^2 > \max_{j \in I_0} \{y_j^2\} - M\delta_2 \\ & \vdots \\ & y_{k+1}^s > \max_{j \in I_0} \{y_j^s\} - M\delta_s \\ & \sum_{i=1}^{s} \delta_i \leqslant s - 1 \\ & \delta_i \in \{0,1\}, i = 1, \ldots, s. \end{aligned} \tag{3.3}$$

Therefore, to obtain another non-dominated vector of problem (2.2) we consider the following model:

$$\begin{aligned}
\theta_1^* = \max \sum_{r=1}^{s} & C_r W \\
\text{s.t. } & A_i W \leqslant b_i, && i=1,\ldots,m \\
& \sum_{r=1}^{s} C_r W \leqslant -\theta_0^* - \varepsilon \\
& C_r W \geqslant \max_{j\in I_0}\{y_j^r\} + \alpha - M\delta_r, && r=1,\ldots,s \\
& \sum_{i=1}^{s} \delta_i \leqslant s-1 \\
& \delta_r \in \{0,1\}, W \geqslant 0, w_j \in Z, j \in J, r=1,\ldots,s
\end{aligned} \tag{3.4}$$

where α is a real positive number and is determined by decision maker, and ε is a very small positive number. This leads to a suitable dispersal of the elements of WDSND vectors of MOMILP problem. If $\delta_r = 1$, then the constraint $C_r W \geqslant \max_{j\in I_0}\{y_j^r\} + \alpha - M\delta_r$ is redundant. The constraint $\sum_{i=1}^{s} \delta_i \leqslant s-1$ implies that at least one of the constraints $C_r W \geqslant \max_{j\in I_0}\{y_j^r\} + \alpha - M\delta_r,\ r = 1,2,\ldots,s$ is not redundant [7].

Let $W_{k+1}^*, W_{k+2}^*, \ldots, W_{k+q}^*$ be the optimal solutions of problem (3.4). Using $I_1 = I_o \cup \{k+1,\ldots,k+q\} = \{1,\ldots,k,k+1,\ldots,k+q\}$ we can obtain another non-dominated vector of problem (2.2). Therefore,

$$\begin{aligned}
\theta_2^* = \max \sum_{r=1}^{s} & C_r W \\
\text{s.t. } & A_i W \leqslant b_i, && i=1,\ldots,m \\
& \sum_{r=1}^{s} C_r W \leqslant -\theta_0^* - \varepsilon \\
& C_r W \geqslant \max_{j\in I_0}\{y_j^r\} + \alpha - M\delta_r, && r=1,\ldots,s \\
& \sum_{i=1}^{s} \delta_i \leqslant s-1 \\
& \delta_r \in \{0,1\}, W \geqslant 0, w_j \in Z, j \in J, r=1,\ldots,s.
\end{aligned} \tag{3.5}$$

Theorem 3: *The optimal solutions of problem (3.4) are the non-dominated vectors of problem (2.2).*

Proof: The proof is similar to that of Theorem 4 in [7] and is not repeated here. □

3.1 *The Modified Algorithm*

the Following Algorithm is a Modified Version of the Proposed algorithm in [7].

Step 0: Solve the problem (3.2) and suppose that I_0 is the indices set of its optimal solutions,

Step 1: Solve the following model:

$$
\begin{aligned}
\theta_q^* = \max \ & \sum_{r=1}^{s} C_r W \\
\text{s.t. } & A_i W \leqslant b_i, && i = 1, \ldots, m \\
& \sum_{r=1}^{s} C_r W \leqslant \theta_{q-1}^* - \varepsilon \\
& C_r W \geqslant \max_{j \in I_q} \{y_j^r\} + \alpha - M\delta_r, && r = 1, \ldots, s \\
& \sum_{i=1}^{s} \delta_i \leqslant s - 1 \\
& \delta_r, \in \{0, 1\}, W \geqslant 0, w_j \in Z, j \in J, r = 1, \ldots, s
\end{aligned}
\tag{3.6}
$$

and suppose that F_q is the set indices of the optimal solutions of model (3.6),
Step 2: If F_q is empty, stop. Otherwise, put $I_q = I_{q-1} \cup F_{q-1}, q = 1, 2, \ldots$, where $F_0 = \phi$, and go to step 1.

Theorem 4: *Optimal solutions of problem (3.6) are non-dominated vectors of problem (2.2).*

Proof: The proof is similar to that of Theorem 4 and is omitted. □

Let $E_q = \{W_1^*, W_2^*, \ldots, W_d^*\}$ be the set of non-dominated vectors of problem (2.2) which have been generated by iterations 1 trough q of the modified algorithm.

Theorem 5: *The proposed algorithm is convergent.*

Proof: The feasible region of problem (2.2) is bounded. Therefore, $\theta_{q-1}^* - \varepsilon < +\infty$ and the proposed algorithm is convergent. □

4. Numerical Example

Consider the following MOMILP problem [6]:

$$
\begin{aligned}
\max \quad & w_1 \\
\max \quad & w_2 \\
\text{s.t.} \quad & w_1 + 2w_2 + 2w_3 \leqslant 4 \\
& 2w_1 + w_2 - 2w_3 \leqslant 2 \\
& w_1, w_2 \geqslant 0, w_3 \in \{0, 1\}.
\end{aligned}
$$

Let $\varepsilon = 0.10^{-6}$ and $\alpha = 0.2$. Using the modified algorithm a WDSND vectors of MOMILP problem is obtained as follows:

$$
\{(0, 2, 0), (2, 0, 1), (1.6, 0.2, 1), (0.2, 1.6, 0), (1.2, 0.4, 1), (0.4, 1.2, 0), (0.8, 0.6, 1)\}.
$$

5. Conclusion

This paper modified the proposed algorithm in [7] to find a well-dispersed subset of non-dominated vectors of an MOMILP problem. In each iteration, the proposed method modifies only the right hand side of some constraints of a single objective.

Hence, the proposed algorithm reduces computational efforts for solving an MOMILP problem.

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