*International Journal of*
*Mathematical Modelling & Computations*
Vol. 05, No. 03, Summer 2015, 203- 217

# Application of Differential Transform Method to Solve Hybrid Fuzzy Differential Equations

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**Abstract.** In this paper, we study the numerical solution of hybrid fuzzy differential equations by using differential transformation method (DTM). This is powerful method which consider the approximate solution of a nonlinear equation as an infinite series usually converging to the accurate solution. Several numerical examples are given and by comparing the numerical results obtained from DTM and Predictor corrector method (PCM), we have studied their accuracy.

Received: 5 August 2014, Revised: 22 March 2015, Accepted: 3 May 2015.

**Keywords:** Hybrid systems, Fuzzy differential equations, Predictor corrector method, Differential transformation method.

**Index to information contained in this paper**



## 1. Introduction

Hybrid systems are devoted to modeling, designing and validating of interactive systems of computer programs and continuous systems. That is control systems that are capable of controlling complex systems which have discrete event dynamics as well as continuous time dynamics which can be modeled by hybrid systems. The differential systems containing fuzzy valued functions and interaction with a discrete time controller are named as hybrid fuzzy differential systems (HFDEs).

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Several numerical techniques has been applied to solve hybrid fuzzy differential equations. For example, Pederson and Sambandham have investigated the numerical solution of this equations by using the Euler and Runge-Kutta methods [7, 8]. Prakash and Kalaiselvi have studied the predictor-corrector method for hybrid fuzzy differential equations [9]. Also, Fard and Bidgoli have solved HFDEs by Nystrom method [3]. Paripour et al. [6] presented a numerical solution for solving hybrid fuzzy differential equations by Adomian decomposition method.
In this study, we develop numerical method for hybrid fuzzy differential equations by an application of Differential transformation method. Differential transform method is one of the analytical method for differential equations. The basic idea was initially introduced by Zhou [11] in 1986. In [4], the authors gave one of the applications of differential transformation method for solving fuzzy fractional Heat equations. Its main application, therein, is to solve both linear and nonlinear initial value problems in electrical circuit analysis. The structure of paper is organized as follows:

In Section 2, we provide some basic definitions to fuzzy valued functions. Section 3, contains a brief review of the hybrid fuzzy differential equations. In Sections 4, hybrid fuzzy differential equations are solved by DTM. Finally, in section 5, we present two examples to check the accuracy of the method and we compare it with predictor corrector method.

## 2. Preliminaries

**Definition 2.1.** [10] A fuzzy number $u$ is a fuzzy subset of the real line with a normal, convex and upper semicontinuous membership function of bounded support. The family of fuzzy numbers will be denoted by $E$. An arbitrary fuzzy number is represented by an ordered pair of functions $(\underline{u}(\alpha), \overline{u}(\alpha))$, $0 \leqslant \alpha \leqslant 1$, that satisfies the following requirements:

– $\underline{u}(\alpha)$ is a bounded left continuous nondecreasing function over $[0, 1]$, with respect to any $\alpha$,

– $\overline{u}(\alpha)$ is a bounded left continuous nonincreasing function over $[0, 1]$, with respect to any $\alpha$,

– $\underline{u}(\alpha) \leqslant \overline{u}(\alpha)$, $0 \leqslant \alpha \leqslant 1$.

Then $\alpha$- level set

$$[u]^\alpha = \{x \in \mathbb{R} : u(x) \geqslant \alpha\},$$

is a closed bounded interval, denoted

$$[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha].$$

**Definition 2.2.** The supremum metric, the space $d_\infty$ on $E$ is defined by

$$d_\infty(u, v) = sup_{0 \leqslant \alpha \leqslant 1} d_H([U]^\alpha, [V]^\alpha),$$

and $(E, d_\infty)$ is a complete metric space.

**Definition 2.3.** A mapping $F : T \to E$ is Hukuhara differentiable at $t_0 \in T \in \mathbb{R}$, if for some $h_0 > 0$, Hukuhara differences $F(t_0+\triangle t) \sim_h F(t_0)$ and $F(t_0) \sim_h F(t_0-\triangle t)$ exist in $E$, for

$$\lim_{\triangle t\to 0^+} d_\infty\big(\frac{F(t_0+\triangle t)\sim_h F(t_0)}{\triangle t}, F'(t_0)\big) = 0,$$

and

$$\lim_{\triangle t\to 0^+} d_\infty\big(\frac{F(t_0)\sim_h F(t_0-\triangle t)}{\triangle t}, F'(t_0)\big) = 0.$$

The fuzzy set $F'(t_0)$ is called the Hukuhara derivative of $F$ at $t_0$.

**Theorem 2.1.** Let $F : \mathbb{R} \longrightarrow \mathbb{E}$ be a function and set $F(t) = (\underline{F}(t,\alpha), \overline{F}(t,\alpha))$, for each $\alpha \in [0, 1]$. Then

– If $F$ is differentiable in the first form $(i)$, then $\underline{F}(t,\alpha)$ and $\overline{F}(t,\alpha)$ are differentiable functions and
$F'(t) = (\underline{F}'(t,\alpha), \overline{F}'(t,\alpha))$.
– If $F$ is differentiable in the second form $(ii)$, then $\underline{F}(t,\alpha)$ and $\overline{F}(t,\alpha)$ are differentiable functions and
$F'(t) = (\overline{F}'(t,\alpha), \underline{F}'(t,\alpha))$.

**Definition 2.4.** Triangular fuzzy number is a fuzzy set $u$ in $E$ that is characterized by an ordered triple $(u^l, u^c, u^r) \in \mathbb{R}^3$ with $u^l \leqslant u^c \leqslant u^r$ such that $[u]^0 = [u^l, u^r]$ and $[u]^1 = \{u^c\}$.
the $\alpha$- level set

$$[u]^\alpha = [u^c - (1-\alpha)(u^c - u^l), u^c + (1-\alpha)(u^r - u^c)],$$

for any $\alpha \in I = [0, 1]$.

**Remark 2.1.** Note that by the above definition, a fuzzy-valued function is $(i)$-differentiable (or $(ii)$-differentiable) of order $n$ if for $s = 1, \dots, n,\ f^{(s)}$ is $(i)$-differentiable (or $(ii)$-differentiable). It is possible that the different orders have different types ($i$ or $ii$) of differentiability, but we do not consider this kind of function in this paper.

Following [1], we define a first-order fuzzy differentiable equation by

$$x' = f(t, x(t)),$$

where $x(t) = \big(x^l(t), x^c(t), x^r(t)\big)$ is a fuzzy function of $t$. $f(t, x(t))$ is a fuzzy-valued function and the fuzzy variable $x'(t)$ is the defined derivative of $x(t)$. Given an initial value $x(t_0) = x_0$ is given, we obtain a fuzzy Cauchy problem of the first-order

$$x' = f(t, x(t)), \quad x(t_0) = x_0.$$

So, considering derivatives of type $(i)$ or $(ii)$, we may replace the fuzzy initial value problem (FIVP) by the equivalent system

$$\begin{aligned} x^{l'}(t) &= f_l\big(t, x^l(t)\big), \quad x^l(t_0) = x_0^l, \\ x^{c'}(t) &= f_c\big(t, x^c(t)\big), \quad x^c(t_0) = x_0^c, \\ x^{r'}(t) &= f_r\big(t, x^r(t)\big), \quad x^r(t_0) = x_0^r, \end{aligned}$$

or

$$\begin{aligned} x^{l'}(t) &= f^r\big(t, x^r(t)\big), \quad x^l(t_0) = x_0^l, \\ x^{c'}(t) &= f^c\big(t, x^c(t)\big), \quad x^c(t_0) = x_0^c, \\ x^{r'}(t) &= f^l\big(t, x^l(t)\big), \quad x^r(t_0) = x_0^r, \end{aligned}$$

the system represents an ordinary Cauchy problem, to which any convergent classical numerical procedure can be applied.

## 3. The Hybrid Fuzzy Differential System

Consider the hybrid fuzzy differential system

$$\begin{cases} x'(t) = f\big(t, x(t), \lambda_j(x_j)\big), & t \in [t_j, t_{j+1}], \\ x(t_j) = x_j, \end{cases} \tag{1}$$

where $0 \leqslant t_0 < t_1 < \cdots < t_j < \cdots, \quad t_j \to \infty, \quad f \in C[\mathbb{R}^+ \times E \times E, E], \quad \lambda_j \in C[E, E]$.

To be specific, the system would look like:

$$x'(t) = \begin{cases} x_0'(t) = f\big(t, x_0(t), \lambda_0(x_0)\big),\ x_0(t_0) = x_0, & t_0 \leqslant t \leqslant t_1, \\ x_1'(t) = f\left(t, x_1(t), \lambda_1(x_1)\right),\ x_1(t_1) = x_1, & t_1 \leqslant t \leqslant t_2, \\ \qquad\vdots \\ x_j'(t) = f\left(t, x_j(t), \lambda_j(x_j)\right),\ x_j(t_j) = x_j, & t_j \leqslant t \leqslant t_{j+1}, \\ \qquad\vdots \end{cases}$$

Assuming that the existence and uniqueness of solutions of (1) hold for each $[t_j, t_{j+1}]$, by the solution of (1) we obtain the following function:

$$x(t) = x(t, t_0, x_0) = \begin{cases} x_0(t), & t_0 \leqslant t \leqslant t_1, \\ x_1(t), & t_1 \leqslant t \leqslant t_2, \\ \ \vdots \\ x_j(t), & t_j \leqslant t \leqslant t_{j+1}, \\ \ \vdots \end{cases}$$

We note that the solutions of (1) are piecewise differentiable in each interval for $t \in [t_j, t_{j+1}]$, for a fixed $x_j \in E$ and $j = 0, 1, 2, \ldots$.

## 4. Differential Transform Method for a Hybrid Fuzzy Differential System

The basic definitions and fundamental operations of differential transform are given in [2]. In this section, we apply DTM to solve equation (1).

**Definition 1.3.** If $x(t)$ is strongly generalized differentiable of order $k$ in the time domain $T$ then If $f$ is $(i)$-differentiable,

$$\varphi^l(t,k) = \frac{d^k(x^l(t))}{dt^k}, \quad \forall t \in T,$$

$$X_i^l(k) = \varphi(t_i,k) = \frac{d^k(x^l(t))}{dt^k}\Bigg]_{t=t_i}, \quad \forall k \in K,$$

$$\varphi^c(t,k) = \frac{d^k(x^c(t))}{dt^k}, \quad \forall t \in T,$$

$$X_i^c(k) = \varphi(t_i,k) = \frac{d^k(x^c(t))}{dt^k}\Bigg]_{t=t_i}, \quad \forall k \in K$$

$$\varphi^r(t,k) = \frac{d^k(x^r(t))}{dt^k}, \quad \forall t \in T,$$

$$X_i^r(k) = \varphi(t_i,k) = \frac{d^k(x^r(t))}{dt^k}\Bigg]_{t=t_i}, \quad \forall k \in K,$$

and if $f$ is $(ii)$-differentiable,

$$\varphi^l(t,k) = \frac{d^k(x^r(t))}{dt^k}, \quad \forall t \in T,$$

$$X_i^l(k) = \varphi(t_i,k) = \frac{d^k(x^r(t))}{dt^k}\Bigg]_{t=t_i}, \quad \forall k \ is \ odd,$$

$$\varphi^c(t,k) = \frac{d^k(x^c(t))}{dt^k}, \quad \forall t \in T,$$

$$X_i^c(k) = \varphi(t_i,k) = \frac{d^k(x^c(t))}{dt^k}\Bigg]_{t=t_i}, \quad \forall k \ is \ odd,$$

$$\varphi^r(t,k) = \frac{d^k(x^l(t))}{dt^k}, \quad \forall t \in T,$$

$$X_i^r(k) = \varphi(t_i,k) = \frac{d^k(x^l(t))}{dt^k}\Bigg]_{t=t_i}, \quad \forall k \ is \ odd,$$

So, if f is $(i)$-differentiable, then $x(t)$ can be represented as

$$x^l(t) = \sum_{k=0}^{\infty} \frac{(t-t_i)^k}{k!} X^l(k),$$

$$x^c(t) = \sum_{k=0}^{\infty} \frac{(t-t_i)^k}{k!} X^c(k),$$

$$x^r(t) = \sum_{k=0}^{\infty} \frac{(t-t_i)^k}{k!} X^r(k),$$

or if $f$ is $(ii)$-differentiable, as

$$\begin{aligned} x^l(t) &= \sum_{k=1,odd}^{\infty} \frac{(t-t_i)^k}{k!} X^r(k) + \sum_{k=0,even}^{\infty} \frac{(t-t_i)^k}{k!} X^l(k), \\ x^c(t) &= \sum_{k=1,odd}^{\infty} \frac{(t-t_i)^k}{k!} X^c(k) + \sum_{k=0,even}^{\infty} \frac{(t-t_i)^k}{k!} X^c(k), \\ x^r(t) &= \sum_{k=1,odd}^{\infty} \frac{(t-t_i)^k}{k!} X^l(k) + \sum_{k=0,even}^{\infty} \frac{(t-t_i)^k}{k!} X^r(k), \end{aligned} \tag{2}$$

The above set of equations is known as the inverse transformation of $X(k)$.

From definition 1.3, it can be proven that the transformation function has the basic mathematical properties shown in table 1.

By using Differential transform formulas in table 1 to solve equation (1), for each interval $[t_j, t_{j+1}]$, $j = 0, 1, 2, \ldots$ we obtain the following relation

$$\begin{aligned} (k+1)X^l(k+1) &= F^l\big(t, X^l(k), \lambda_j(x_j)\big) \\ (k+1)X^c(k+1) &= F^c\big(t, X^c(k), \lambda_j(x_j)\big) \\ (k+1)X^r(k+1) &= F^r\big(t, X^r(k), \lambda_j(x_j)\big) \quad \forall k \textit{ is even}, \end{aligned} \tag{3}$$

or

$$\begin{aligned} (k+1)X^l(k+1) &= F^r\big(t, X^r(k), \lambda_j(x_j)\big) \\ (k+1)X^c(k+1) &= F^c\big(t, X^c(k), \lambda_j(x_j)\big) \\ (k+1)X^r(k+1) &= F^l\big(t, X^l(k), \lambda_j(x_j)\big) \quad \forall k \textit{ is odd}, \end{aligned} \tag{4}$$

where $F$ indicates differential transform $f$. From the initial condition given by Eq. (1) we have

$$X^l(0) = x^l(t_0), \quad X^c(0) = x^c(t_0), \quad X^r(0) = x^r(t_0) \tag{5}$$

Substituting Eq. (5) into (4) and (3), we get

$$X^l(k), X^c(k), X^r(k)$$

now, with replace $X^l(k), X^c(k), X^r(k)$ in Eq. (2), will be obtained numerical solution which is convergent to exact solution.

Table 1. The fundamental mathematical operations.

| Original function | Transformed function |
|---|---|
| $u(x)=f(x)\pm g(x)$ | $U(k)=F(k)\pm G(k)$ |
| $u(x)=\lambda g(x)$ | $U(k)=\lambda G(k)$ |
| $u(x)=\dfrac{\partial g(x)}{\partial x}$ | $U(k)=(k+1)G(k+1)$ |
| $u(x)=\dfrac{\partial^m g(x)}{\partial x^m}$ | $U(k)=(k+1)\ldots(k+m)G(k+m)$ |
| $u(x)=x^m$ | $U(k)=\delta(k-m)=\begin{cases}1 & k=m,\\ 0 & otherwise\end{cases}$ |
| $u(x)=f(x)g(x)$ | $U(k)=\sum_{r=0}^{k}F(r)G(k-r)$ |
| $u(x)=\sin(\omega x+\alpha)$ | $U(k)=\dfrac{\omega^k}{k!}\sin(\dfrac{k\pi}{2}+\alpha)$ |
| $u(x)=\cos(\omega x+\alpha)$ | $U(k)=\dfrac{\omega^k}{k!}\cos(\dfrac{k\pi}{2}+\alpha)$ |

## 5. Numerical Examples

Here, we consider two examples to illustrate the DTM. Their computations have been carried out using MATLAB.

**Example 6.1.** Consider the initial value problem,

$$\begin{cases} x'(t)=x(t)+m(t)\lambda_k(x_{t_k}), \quad t\in[t_k,t_{k+1}], \quad t_k=k, \quad k=0,1,\ldots, \\ x(0)=[0.75,1,1.125], \end{cases} \tag{6}$$

where

$$m(t)=\begin{cases} 2\big(t(mod\ 1)\big), & t(mod\ 1)\leqslant 0.5, \\ 2\big(1-t(mod\ 1)\big), & t(mod\ 1)>0.5, \end{cases} \tag{7}$$

and

$$\lambda_k(\mu)=\begin{cases} \widehat{0},\ k=0, \\ \mu,\ k\in\{1,2,\ldots\}, \end{cases} \tag{8}$$

for which $\widehat{0}\in E^n$ is defined as $\widehat{0}(x)=1$ if $x=0$ and $\widehat{0}(x)=0$ if $x\neq 0$.
The hybrid fuzzy initial problem (6) is equivalent to the following system of fuzzy initial value problems:

$$\begin{cases} x_0'(t)=x_0(t), \qquad t\in[0,1], \\ x(0)=[0.75,1,1.125], \\ x_i'(t)=x_i(t)+m(t)x_i(t_i), \quad t\in[t_i,t_{i+1}]. \end{cases}$$

In Eq. (6), $x(t)+m(t)\lambda_k(x_{t_k})$ is a continuous function of $t$, $x$ and $\lambda_k(x_{t_k})$. Therefore, by Example 6.1 of Kaleva [5], for each $k = 0, 1, 2, \ldots$, the fuzzy initial value problem

$$\begin{cases} x'(t) = f\big(t, x(t), \lambda_k(x_k)\big), & t \in [t_k, t_{k+1}], \\ x(t_k) = x_k, \end{cases}$$

has a unique solution on $[t_k, t_{k+1}]$.
For $[0, 1]$, the exact solution of Eq. (6) satisfies,

$$x(t) = [0.75e^t, e^t, 1.125e^t].$$

For $[1, 1.5]$ the exact solution of (6) satisfies,

$$x(t) = x(1)(3e^{t-1} - 2t).$$

For $[1.5, 2]$ the exact solution of (6) satisfies,

$$x(t) = x(1)\big(2t - 2te^{t-1.5}(3\sqrt{e} - 4)\big).$$

Taking the differential transform of (6) for $t \in [0, 1]$ leads to

$$(k+1)X^l(k+1) = X^l(k), \quad (k+1)X^c(k+1) = X^c(k), \quad (k+1)X^r(k+1) = X^r(k),$$

and for $t \in [1, 1.5]$, we have

$$(k+1)X^l(k+1) = X^l(k) + 2(\delta(k-1) - \delta(k))x(1).$$

$$(k+1)X^c(k+1) = X^c(k) + 2(\delta(k-1) - \delta(k))x(1).$$

$$(k+1)X^r(k+1) = X^r(k) + 2(\delta(k-1) - \delta(k))x(1).$$

If we consider the interval $[1.5, 2]$, we have

$$(k+1)X^l(k+1) = X^l(k) + (4\delta(k) - 2\delta(k-1))x(1.5).$$

$$(k+1)X^c(k+1) = X^c(k) + (4\delta(k) - 2\delta(k-1))x(1.5).$$

$$(k+1)X^r(k+1) = X^r(k) + (4\delta(k) - 2\delta(k-1))x(1.5).$$

Using the above process, $x^l, x^c$ and $x^r$ are obtained. to solve this problem, we use only the first ten terms of the differential transform series. The results of Example 6.1 on $[0, 2]$, by using DTM and PCM , are shown in Fig. 1. and the numerical results are shown in Table 2, 3 and 4. The results show that this method is reliable and efficient techniques to find analytic solutions for HFDEs. The approximate solution of this problem by DTM is nearly similar to those obtained by PCM. In this example, the results were in excellent agreement with those of the exact

Table 2. Comparison of approximation solutions with exact solution for $x^l$ of Example 1.

| t | Exact | PCM | DTM |
|---|---|---|---|
| 0.0 | 0.750000 | 0.750000 | 0.750000 |
| 0.1 | 0.828878 | 0.828878 | 0.828878 |
| 0.2 | 0.916052 | 0.916052 | 0.916052 |
| 0.3 | 1.012394 | 1.012396 | 1.012394 |
| 0.4 | 1.118868 | 1.118874 | 1.118868 |
| 0.5 | 1.236540 | 1.236550 | 1.236540 |
| 0.6 | 1.366589 | 1.366602 | 1.366589 |
| 0.7 | 1.510314 | 1.510333 | 1.510314 |
| 0.8 | 1.669155 | 1.669180 | 1.669155 |
| 0.9 | 1.844702 | 1.844734 | 1.844702 |
| 1.0 | 2.038711 | 2.038752 | 2.038711 |
| 1.1 | 2.274208 | 2.290615 | 2.274208 |
| 1.2 | 2.577355 | 2.604161 | 2.577355 |
| 1.3 | 2.955267 | 2.985790 | 2.955266 |
| 1.4 | 3.415808 | 3.442668 | 3.415805 |
| 1.5 | 3.967666 | 3.977376 | 3.967662 |
| 1.6 | 4.554410 | 4.533824 | 4.578274 |
| 1.7 | 5.210226 | 5.113589 | 5.210222 |
| 1.8 | 5.865754 | 5.719217 | 5.865749 |
| 1.9 | 6.547343 | 6.423019 | 6.547337 |
| 2.0 | 7.257731 | 7.295736 | 7.257725 |

solution.

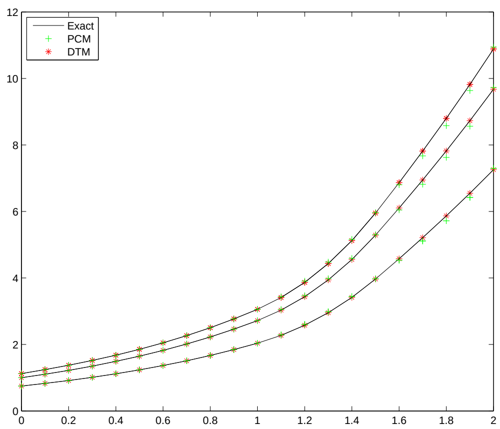


Figure 1. Comparison between the exact solution and behavior of the solution obtained by DTM and PCM of Example 1.

**Example 6.2.** Consider the initial value problem,

$$\begin{cases} x'(t) = -x(t) + m(t)\lambda_k(x_{t_k}), \quad t \in [t_k, t_{k+1}], \quad t_k = k, \quad k = 0, 1, \ldots, \\ x(0) = [0.75, 1, 1.125], \\ \end{cases} \tag{9}$$

Table 3. Comparison of approximation solutions with exact solution for $x^c$ of Example 1.

| t | Exact | PCM | DTM |
|---|---|---|---|
| 0.0 | 1.000000 | 1.000000 | 1.000000 |
| 0.1 | 1.105170 | 1.105170 | 1.105170 |
| 0.2 | 1.221402 | 1.221402 | 1.221402 |
| 0.3 | 1.349858 | 1.349862 | 1.349858 |
| 0.4 | 1.491824 | 1.491832 | 1.491824 |
| 0.5 | 1.648721 | 1.648733 | 1.648721 |
| 0.6 | 1.822118 | 1.822137 | 1.822118 |
| 0.7 | 2.013752 | 2.013777 | 2.013752 |
| 0.8 | 2.225540 | 2.225574 | 2.225540 |
| 0.9 | 2.459603 | 2.459646 | 2.459603 |
| 1.0 | 2.718281 | 2.718336 | 2.718281 |
| 1.1 | 3.032278 | 3.054154 | 3.032277 |
| 1.2 | 3.436474 | 3.472214 | 3.436474 |
| 1.3 | 3.940357 | 3.981054 | 3.940358 |
| 1.4 | 4.554410 | 4.590224 | 4.554413 |
| 1.5 | 5.290221 | 5.303169 | 5.290226 |
| 1.6 | 6.104371 | 6.045098 | 6.104376 |
| 1.7 | 6.946969 | 6.818119 | 6.946974 |
| 1.8 | 7.821006 | 7.625623 | 7.821013 |
| 1.9 | 8.729790 | 8.564025 | 8.729797 |
| 2.0 | 9.676975 | 9.727648 | 9.676983 |

Table 4. Comparison of approximation solutions with exact solution for $x^r$ of Example 1.

| t | Exact | PCM | DTM |
|---|---|---|---|
| 0.0 | 1.125000 | 1.125000 | 1.125000 |
| 0.1 | 1.243317 | 1.243317 | 1.243317 |
| 0.2 | 1.374078 | 1.374078 | 1.374078 |
| 0.3 | 1.518591 | 1.518594 | 1.518591 |
| 0.4 | 1.678302 | 1.678311 | 1.678302 |
| 0.5 | 1.854811 | 1.854825 | 1.854811 |
| 0.6 | 2.049883 | 2.049904 | 2.049883 |
| 0.7 | 2.265471 | 2.265500 | 2.265471 |
| 0.8 | 2.503733 | 2.503771 | 2.503733 |
| 0.9 | 2.767053 | 2.767102 | 2.767053 |
| 1.0 | 3.058067 | 3.058128 | 3.058067 |
| 1.1 | 3.411312 | 3.435923 | 3.411312 |
| 1.2 | 3.866033 | 3.906241 | 3.866034 |
| 1.3 | 4.432901 | 4.478686 | 4.432904 |
| 1.4 | 5.123712 | 5.164003 | 5.123717 |
| 1.5 | 5.951499 | 5.966065 | 5.951508 |
| 1.6 | 6.867417 | 6.800736 | 6.867427 |
| 1.7 | 7.815340 | 7.670384 | 7.815351 |
| 1.8 | 8.798632 | 8.578826 | 8.798644 |
| 1.9 | 9.821014 | 9.634529 | 9.821028 |
| 2.0 | 10.886597 | 10.943604 | 10.886612 |

where

$$m(t) = |\sin(\pi t)|, \quad k = 0, 1, \ldots, \tag{10}$$

$$\lambda_k(\mu) = \begin{cases} \widehat{0}, \ k = 0, \\ \mu, \ k \in \{1, 2, \cdots\}, \end{cases}$$

For $[0,1]$, the exact solution of (9) satisfies,

$$x(t) = [-0.1875e^t + 0.9375e^{-t}, e^{-t}, 0.1875e^t + 0.9375e^{-t}].$$

For $[1,2]$, the exact solution of (9) satisfies,

$$\begin{aligned} x(t) = \Bigg[ & -0.1875\bigg(e^t + \frac{1}{1+\pi^2}\Big(e\big(\sin(\pi t) + \pi\cos(\pi t)\big) + \pi e^t\Big)\bigg) \\ & + 0.9375\bigg(e^{-t} - \frac{1}{1+\pi^2}\Big(e^{-1}\big(\sin(\pi t) - \pi\cos(\pi t)\big) - \pi e^{-t}\Big)\bigg), e^{-t} \\ & - \frac{1}{1+\pi^2}\Big(e^{-1}\big(\sin(\pi t) - \pi\cos(\pi t)\big) - \pi e^{-t}\Big), 0.1875\bigg(e^t \\ & + \frac{1}{1+\pi^2}\Big(e\big(\sin(\pi t) + \pi\cos(\pi t)\big) + \pi e^t\Big)\bigg) \\ & + 0.9375\bigg(e^{-t} - \frac{1}{1+\pi^2}\Big(e^{-1}\big(\sin(\pi t) - \pi\cos(\pi t)\big) - \pi e^{-t}\Big)\bigg)\Bigg]. \end{aligned}$$

The hybrid fuzzy initial value problem (9) is equivalent to the following system:

$$\begin{cases} x_0^{l'}(t) = -x_0^r(t), & t \in [0,1], \\ x_0^{c'}(t) = -x_0^c(t), \\ x_0^{r'}(t) = -x_0^l(t), \\ x(0) = [0.75, 1, 1.125], \\ x_i^{l'}(t) = -x_i^r(t) + m(t)x_i^l(t_i), & t \in [t_i, t_{i+1}], \\ x_i^{c'}(t) = -x_i^c(t) + m(t)x_i^c(t_i), \\ x_i^{r'}(t) = -x_i^l(t) + m(t)x_i^r(t_i). \end{cases} \tag{11}$$

The system is simplified as follows:

$$\begin{cases} x_0^{l''}(t) = x_0^l(t), & t \in [0,1], \\ x_0^{c'}(t) = -x_0^c(t), \\ x_0^{r''}(t) = x_0^r(t), \\ x(0) = [0.75, 1, 1.125], \\ x_i^{l''}(t) = x_i^l(t) + m(t)x_i^l(t_i), & t \in [t_i, t_{i+1}], \\ x_i^{c'}(t) = -x_i^c(t) + m(t)x_i^c(t_i), \\ x_i^{r''}(t) = x_i^r(t) + m(t)x_i^r(t_i). \end{cases} \tag{12}$$

Taking the differential transform of (12) for $x^l$ and $x^r$ $\forall t \in [0,1]$ leads to

$$(k+1)(k+2)X^l(k+2) = X^l(k),$$

$$(k+1)(k+2)X^r(k+2) = X^r(k),$$

and for $t \in [1,2]$, we have

$$(k+1)(k+2)X^l(k+2) = X^l(k) - \frac{\pi^k}{k!}\sin(\frac{k\pi}{2})x(1),$$

$$(k+1)(k+2)X^r(k+2) = X^r(k) - \frac{\pi^k}{k!}\sin(\frac{k\pi}{2})x(1),$$

Also, taking the differential transform of (12) for $x^c, \forall t \in [0,1]$ leads to

$$(k+1)X^c(k+1) = -X^c(k),$$

and for $t \in [1,2]$

$$(k+1)X^c(k+1) = -X^c(k) - \frac{\pi^k}{k!}\sin(\frac{k\pi}{2})x(1),$$

To solve this problem, we use only the first ten terms of the differential transform series.
The comparison between the exact and numerical solutions on $[0,2]$ is shown in Fig. 2 and the numerical results are shown in Table 5, 6 and 7. The approximate solution of this problem by TDM is nearly similar to those obtained with the exact solution, But the results of PCM weren't in excellent agreement with the exact solution. From Fig. 2, it's clear that DTM works better than PCM.

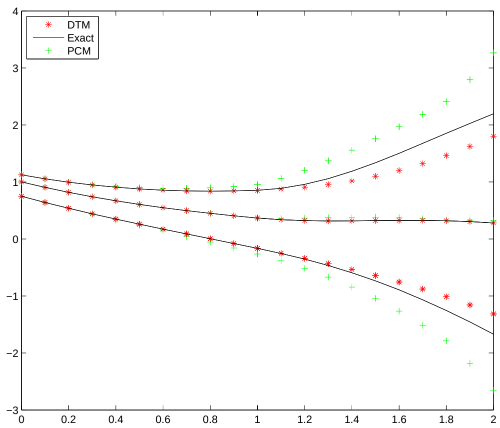


Figure 2. Comparison between the exact solution and behavior of the solution obtained by DTM and PCM of Example 2.

Table 5. Comparison of approximation solutions with exact solution for $x^l$ of Example 2.

| t | Exact | PCM | DTM |
|---|---|---|---|
| 0.0 | 0.750000 | 0.750000 | 0.750000 |
| 0.1 | 0.641065 | 0.641065 | 0.641065 |
| 0.2 | 0.538547 | 0.538547 | 0.538547 |
| 0.3 | 0.441418 | 0.435666 | 0.441418 |
| 0.4 | 0.348707 | 0.335927 | 0.348707 |
| 0.5 | 0.259487 | 0.238110 | 0.259487 |
| 0.6 | 0.172863 | 0.141043 | 0.172863 |
| 0.7 | 0.087970 | 0.043541 | 0.087970 |
| 0.8 | 0.003956 | -0.055610 | 0.003956 |
| 0.9 | -0.080016 | -0.157676 | -0.080016 |
| 1.0 | -0.164790 | -0.263982 | -0.164791 |
| 1.1 | -0.254229 | -0.382524 | -0.251300 |
| 1.2 | -0.353721 | -0.516839 | -0.340830 |
| 1.3 | -0.465896 | -0.670146 | -0.434733 |
| 1.4 | -0.592617 | -0.844844 | -0.534310 |
| 1.5 | -0.734925 | -1.042712 | -0.640790 |
| 1.6 | -0.893071 | -1.264997 | -0.755319 |
| 1.7 | -1.066635 | -1.512581 | -0.878962 |
| 1.8 | -1.254716 | -1.786223 | -1.012720 |
| 1.9 | -1.456183 | -2.183149 | -1.157558 |
| 2.0 | -1.669959 | -2.653593 | -1.314439 |

Table 6. Comparison of approximation solutions with exact solution for $x^c$ of Example 2.

| t | Exact | PCM | DTM |
|---|---|---|---|
| 0.0 | 1.000000 | 1.000000 | 1.000000 |
| 0.1 | 0.904837 | 0.904837 | 0.904837 |
| 0.2 | 0.818730 | 0.818730 | 0.818730 |
| 0.3 | 0.740818 | 0.740822 | 0.740818 |
| 0.4 | 0.670320 | 0.670328 | 0.670320 |
| 0.5 | 0.606530 | 0.606542 | 0.606530 |
| 0.6 | 0.548811 | 0.548825 | 0.548811 |
| 0.7 | 0.496585 | 0.496601 | 0.496585 |
| 0.8 | 0.449328 | 0.449346 | 0.449328 |
| 0.9 | 0.406569 | 0.406587 | 0.406569 |
| 1.0 | 0.367879 | 0.367898 | 0.367879 |
| 1.1 | 0.338415 | 0.363150 | 0.338415 |
| 1.2 | 0.322120 | 0.366600 | 0.322120 |
| 1.3 | 0.316184 | 0.373792 | 0.316184 |
| 1.4 | 0.317201 | 0.380288 | 0.317201 |
| 1.5 | 0.321465 | 0.382036 | 0.321465 |
| 1.6 | 0.325294 | 0.375775 | 0.325296 |
| 1.7 | 0.325361 | 0.359321 | 0.325369 |
| 1.8 | 0.318987 | 0.331754 | 0.319022 |
| 1.9 | 0.304378 | 0.325420 | 0.304508 |
| 2.0 | 0.280777 | 0.3262256 | 0.281206 |

Table 7. Comparison of approximation solutions with exact solution for $x^r$ of Example 2.

| t | Exact | PCM | DTM |
|---|---|---|---|
| 0.0 | 1.125000 | 1.125000 | 1.125000 |
| 0.1 | 1.055504 | 1.055504 | 1.055504 |
| 0.2 | 0.996573 | 0.996573 | 0.996573 |
| 0.3 | 0.947615 | 0.953376 | 0.947615 |
| 0.4 | 0.908142 | 0.920938 | 0.908142 |
| 0.5 | 0.877757 | 0.899155 | 0.877757 |
| 0.6 | 0.856158 | 0.888004 | 0.856158 |
| 0.7 | 0.843127 | 0.887585 | 0.843127 |
| 0.8 | 0.838534 | 0.898134 | 0.838534 |
| 0.9 | 0.842334 | 0.920027 | 0.842334 |
| 1.0 | 0.854564 | 0.953791 | 0.8545645 |
| 1.1 | 0.888758 | 1.063430 | 0.875792 |
| 1.2 | 0.957697 | 1.204215 | 0.908407 |
| 1.3 | 1.058741 | 1.371007 | 0.955100 |
| 1.4 | 1.187370 | 1.557884 | 1.018214 |
| 1.5 | 1.337673 | 1.759031 | 1.099586 |
| 1.6 | 1.502999 | 1.969576 | 1.200445 |
| 1.7 | 1.676689 | 2.186309 | 1.321378 |
| 1.8 | 1.852819 | 2.408264 | 1.462371 |
| 1.9 | 2.026894 | 2.793312 | 1.622898 |
| 2.0 | 2.196416 | 3.265266 | 1.802044 |

## 6. Conclusion

In this paper, differential transform method has been presented for solving hybrid fuzzy differential equations. This method is so powerful and efficient that it give approximations of higher accuracy. Moreover, the DTM, which is based on the Teylor series expansion, constructs an analytical solution in the form of polynomial series solution by means of an iterative procedure. It's rapid convergence show that the method is reliable and introduces a significant improvement in solving the HFDEs. In this method, the accuracy of the obtained solution can be improved by taking more trems in the solution. Two examples were tested by applying the DTM and PCM and the results have shown remarkable performance.

## References


[1] S. Abbasbandy, T. Allahviranloo, Oscar Lopez-Pouso and Juan J. Nieto, *Numerical methods for fuzzy differential inclusions*, Int. J. Comp. Math, **48** (2004) 1633–1641.

[2] J. Biazar and M. Eslami, *Differential transform method for quadratic Riccati differential equation*, Int. J. Nonlin. Sci, **9** (2010) 444–447.

[3] O. S. Fard and T. A. Bidgoli, *The Nystrom method for hybrid fuzzy differential equation IVPs* , J. King Soud University, **23** (2010) 371–379.

[4] B. Ghazanfari and P. Ebrahimi, *Differential Transformation Method For Solving Fuzzy Fractional Heat Equations* , Int. J. Math. Modll. Comput, **5 (1)** (2015) 81–89.

[5] O. Kaleva, *Fuzzy differential equations*, Fuzzy Sets Syst, **24** (1987) 301–317.

[6] M. Paripour, E. Hajilou, A. Hajilou and H. Heidari, *Application of Adomian decomposition method to solve hybrid fuzzy differential equations*, Journal of Taibah University for Science **9** (2015) 95–103.

[7] S. Pederson and M. Sambandham, *Numerical solution to hybrid fuzzy systems*, Math. Comput. Model, **45** (2007) 1133–1144.

[8] S. Pederson and M. Sambandham, *The Runge-Kutta method for hybrid fuzzy differential equation*, Nonlinear Anal. Hybrid Syst, **2** (2008) 626–634.

[9] P. Prakash and V. Kalaiselvi, *Numerical solution of hybrid fuzzy differential equations by predictor-corrector method*, Int. J. Comp. Math, **86** (2009) 121–134.

[10] C. X. Wu and M. Ma, *Embedding problem of fuzzy number space: part 1.*, Fuzzy Sets Syst, **44** (1991) 33–38.

[11] J. K. Zhou, *Differential Transformation and its Application for Electrical Circuits*, Huazhong University Press, Wuhan China, (1986).