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An $M/G/1$ Queue with Regular and Optional Phase Vacation and with State Dependent Arrival Rate

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Abstract. We consider an $M/G/1$ queue with regular and optional phase vacation and with state dependent arrival rate. The vacation policy is after completion of service if there are no customers in the system, the server takes vacation consisting of K-phases, each phase is generally distributed. Here the first phase is compulsory where as the other phases are optional. For this model the supplementary variable technique has been applied to obtain the probability generating functions of number of customers in the queue at the different server states. Some particular models are obtained and a numerical study is also carried out.

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1. Introduction

In the $M/G/1$ queueing system, the concept of vacation had been first studied by [11]. They introduced the concept of modified service time which has a main role in the system with general service and vacation times. In many examples such as production systems, bank services, computer and communication networks, these systems have the concept of vacation. For overhauling or maintanance of the system the server may go to (system) a vacation.

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The classic $M/G/1$ vacation queues with various vacation policies have been well studied (see [3, 4], [5] and [8]). [2] considered a $GI/M/1$ queue with phase type of working vacations and vacation interruption where the vacation time follows a phase type distribution. Two monographes of [13, 16] also collected the research results of the classical $M/G/1$ vacation queues. [14, 15] discussed the discrete time $GI/Geo/1$ queue with server vacations and the $GI/M/1$ queue with PH vacations or setup times, respectively.

Several authors studied the queueing system with heterogeneous arrival and heterogeneous service. [10] studied the characteristics for the heterogeneous batch arrival queue with server startup and breakdown and they obtained the steady state behavior of the system size distribution at stationary point of time as well as the queue size distribution at departure point of time. Later [9] made the contribution to the control policy of $M/G/1$ queue with server vacations, startup and breakdowns. The system characteristics of such a model are analyzed by the author and author obtained total expected cost function per unit time was developed to determine the optimal threshold of N policies at a minimum cost. [7] have analyzed a single server Bernoulli vacation queue with two type of services and with restricted admissibillity. [7] have studied finite population single server batch service queue with compulsory server vacation. [1] considered a $MAP/M/c$ queueing system in which group of servers take a simultaneous phase type vacation and he obtained the steady state solutions. [12] obtained the joint distributions of the length of a busy period, the number of customers served during the busy period and the residual interarrival time at the instant the busy period ends for a $GI/M/1$ queue with multiple phase type vacations.

As an example consider the optical access network with multiple wavelengths, the data arrive randomly and are transmitted by wave lengths. In networks, wave lengths are distributed and most of them transmit the data normally, and the left wavelengths stay in Optical Network Unit (ONU) and not transmit the data. When there are more data, the left wavelengths also can be used to transmit data during a certain period, and the transmit process can be operated in multiple phases. In this paper we consider an $M/G/1$ queue with regular and optional phase vacation and with state dependent arrival rate. The vacation policy is after completion of service if there are no customers in the system, the server takes vacation consisting of K-phases, each phase is generally distributed. Here the first phase is compulsory where as the other phases are optional. We give the mathematical description and analysis in section 2. Section 3 deals with some particular models and section 4 presents some numerical results related to the model analyzed in this paper. The last section gives a conclusion.

2. The Mathematical Model and Analysis

A single server queueing system has been considered. For this model, the arrival follows Poisson with parameter λ, service time is generally distributed with distribution function $B(x)$ whose Laplace stieltjes transform(LST) is $B^*(s)$. After completion of service if there are no customers in the system, the server takes vacation consisting of K-phases with each phase has time duration $V_1, V_2, V_3, ..., V_K$ all are independent random variables with distribution functions$V_i(x), i = 1, 2, ..., K$. Here phase one is compulsory after completion of phase i the server takes phase $i+1$ with probability $\theta_i, i = 0, 1, ..., K(\theta_0 = 1, \theta_K = 0)$. After completion of vacation the server enter in to the system independent of the number of customers in the system. That is, if there are customers in the queue the server starts service using FCFS rule. Otherwise the server waits idle for a new arrival. The arriving customers waiting in a queue of infinite capacity, if the service is not immediate. The modified vacation period is

$$\left.\begin{aligned} V &= V_1 \text{ with probability } (1-\theta_1) \\ &= V_1 + V_2 \text{ with probability } \theta_1(1-\theta_2) \\ &\dots \\ &= V_1 + V_2 + ... + V_{K-1} \text{ with probability } \theta_1\theta_2...\theta_{K-2}(1-\theta_{K-1}) \\ &= V_1 + V_2 + ... + V_K \text{with probability } \theta_1\theta_2...\theta_{K-1} \end{aligned}\right\} \quad (1)$$

and the LST of V is

$$V^*(s) = \sum_{j=1}^{K}\Big(\prod_{i=1}^{j}\theta_{i-1}V_i^*(s)\Big)(1-\theta_j) \quad (2)$$

with

$$E(V) = \sum_{j=1}^{K}\Big(\prod_{i=1}^{j}\theta_{i-1}\Big)E(V_j) \quad (3)$$

The arrival rate have the following definitions

$$i.e.\ \lambda = \begin{cases} \lambda_0, & \text{if the arrival is during idle period} \\ \lambda_1, & \text{if the arrival is during service period} \\ \lambda_2, & \text{if the arrival is during vacation period} \end{cases}$$

The time required by a customer to complete a service cycle is$B_c = B + V$ where V is defined in equation (1). The LST of B_c is $B_c^*(s)$, $B_c^*(s) = B^*(s)V^*(s)$, where$V^*(s)$ is given in equation (2) $E(B_c) = E(s) + E(v)$,where $E(v)$is given in equation (3)

Asssume $B(0) = V_i(0) = 0$,$B(\infty) = V_i(\infty) = 1, i = 1, 2, ..., K$. The elapsed service time at time t is defined by $\xi_0(t)$ and the elapsed vacation time of phase i is denoted by $\eta_i(t)$.

Let $Y(t)$ be the state of the server at time t

$$i.e.\ Y(t) = \begin{cases} 0, & \text{if the server is idle at time t} \\ i, & \text{if the server is } i^{th} \text{ phase of vacation at time t} \\ K+1, & \text{if the server is busy at time t} \end{cases}$$

Let the random variable $L(t)$ is defined as

$$L(t) = \begin{cases} 0, & \text{if } Y(t) = 0 \\ \eta_i(t), & \text{if } Y(t) = i, i = 1, 2, .., K \\ \xi_0(t), & \text{if } Y(t) = K+1 \end{cases}$$

and let N(t) be the number of customers in the queue. Now we define the probabilities

$$Q(t) = Pr\{N(t) = 0, L(t) = 0\}$$

$$P_n(t,x)dx = Pr\left\{N(t)=n, Y(t)=K+1, x<\xi_0(t)\leqslant x+dx\right\}, n\geqslant 0$$

$$R_{i,n}(t,x)dx = Pr\left\{N(t)=n, Y(t)=i, x<\eta_i(t)\leqslant x+dx\right\}, n\geqslant 0, i=1,2,...,K$$

where $\{(N(t),Y(t));t\geqslant 0\}$ is a Bivariate Markov process with state space $S = \{(0,0)\}\cup\{(K+1,j)\}\cup\{(i,j)\}, i=1,2,...,K, j\geqslant 0$.

The hazard rate function of B is $\mu(x)dx = \dfrac{dB(x)}{1-B(x)}$, is the conditional probability of completion of a service during the time interval $(x,x+dx]$ given that elpased service time is x. The similar quantity for V_i is $\eta_i(x)dx = \dfrac{dV_i(x)}{1-V_i(x)}, i=1,2,...,K$.

In steady state, the probabilities are $Q=\lim_{t\to\infty}Q(t)$, $P_n(x)=\lim_{t\to\infty}P_n(t,x)$ and $R_{i,n}(x)=\lim_{t\to\infty}R_{i,n}(t,x)$.

The model is governed by the following differential difference equations:

$$\frac{d}{dx}P_0(x)+(\lambda_1+\mu(x))P_0(x)=0 \tag{4}$$

$$\frac{d}{dx}P_n(x)+(\lambda_1+\mu(x))P_n(x)=\lambda_1 P_{n-1}(x), n\geqslant 0, x>0 \tag{5}$$

$$\frac{d}{dx}R_{i,0}(x)+(\lambda_2+\eta_i(x))R_{i,0}(x)=0, i=1,2,...,K \tag{6}$$

$$\frac{d}{dx}R_{i,n}(x)+(\lambda_2+\eta_i(x))R_{i,n}(x)=\lambda_2 R_{i,n-1}(x), i=1,2,...,K \tag{7}$$

The boundary conditions are

$$\lambda_0 Q=\sum_{i=1}^{K-1}(1-\theta_i)\int_0^\infty \eta_i(x)R_{i,0}(x)dx+\int_0^\infty \eta_K(x)R_{K,0}(x)dx \tag{8}$$

$$P_0(0)=\lambda_0 Q+\sum_{i=1}^{K}(1-\theta_i)\int_0^\infty \eta_i(x)R_{i,1}(x)dx+\int_0^\infty P_1(x)\mu(x)dx \tag{9}$$

$$P_n(0)=\sum_{i=1}^{K}(1-\theta_i)\int_0^\infty \eta_i(x)R_{i,n+1}(x)dx+\int_0^\infty P_{n+1}(x)\mu(x)dx, n\geqslant 1 \tag{10}$$

$$R_{1,0}(0)=\int_0^\infty \mu(x)P_0(x)dx \tag{11}$$

$$R_{1,n}(0)=0, n\geqslant 1 \tag{12}$$

$$R_{i,n}(0) = \theta_{i-1} \int_0^\infty R_{i-1,n}(x)\eta_{i-1}(x)dx, i = 2, 3, ..., K; n = 0, 1, 2, \cdots \tag{13}$$

The normalization condition is

$$Q + P(1) + \sum_{i=1}^{K} R_i(1) = 1$$

For the analysis, the following probability generating functions have been defined

$$P(x, z) = \sum_{n=0}^{\infty} z^n P_n(x)$$

and

$$R_i(x, z) = \sum_{n=0}^{\infty} z^n R_{i,n}(x), i = 1, 2, ..., K$$

From equation (4), we have

$$P_0(x) = P_0(0)(1 - B(x))e^{-\lambda_1 x} \tag{14}$$

Multiplying equation (5) by z^n, summing from 1 to ∞ and adding equation (4), we get

$$P(x, z) = P(0, z)(1 - B(x))e^{-\lambda_1(1-z)x} \tag{15}$$

Multiplying equation (7) by z^n, summing from 1 to ∞ and adding equation (6), we get

$$R_i(x, z) = R_i(0, z)(1 - V_i(x))e^{-\lambda_2(1-z)x}, i = 1, 2, ..., K \tag{16}$$

From equation (6), we get

$$R_{i,0}(x) = R_{i,0}(0)(1 - V_i(x))e^{-\lambda_2 x}, i = 1, 2, ..., K \tag{17}$$

Multiplying equation (10) byz^n, summing from 1 to ∞, adding equation (9) and multiply by z, we get

$$\begin{aligned} zP(0, z) = z\lambda_0 Q + \sum_{i=1}^{k}(1 - \theta_i)\Big[\int_0^\infty \eta_i(x)R_i(x, z)dx - \int_0^\infty \eta_i(x)R_{i,0}(x)dx\Big] \\ + \int_0^\infty \mu(x)P(x, z)dx - \int_0^\infty \mu(x)P_0(x)dx, i = 1, 2, ..., K \end{aligned} \tag{18}$$

From equation (17),we have

$$\int_0^\infty R_{i,0}(x)\eta_i(x)dx = R_{i,0}(0)V_i^*(\lambda_2), i = 1, 2, ..., K \tag{19}$$

From equation (16), we have

$$\int_0^\infty R_i(x,z)\eta_i(x)dx = R_i(0,z)V_i^*(\lambda_2(1-z)), \tag{20}$$

From equation (14), we have

$$\int_0^\infty P_0(x)\mu(x)dx = P_0(0)B^*(\lambda_1) \tag{21}$$

From equation (15), we have

$$\int_0^\infty P(x,z)\mu(x)dx = P(0,z)B^*(\lambda_1(1-z)) \tag{22}$$

Using equations (19),(20),(21) and (22) in (18),

$$\left.\begin{aligned} P(0,z) &= \lambda_0 zQ + \sum_{i=1}^{K}(1-\theta_i)R_i(0,z)V_i^*(\lambda_2(1-z)) \\ &\quad - \sum_{i=1}^{K}(1-\theta_i)R_{i,0}(0)V_i^*(\lambda_2) - P_0(0)B^*(\lambda_1) \end{aligned}\right\} \tag{23}$$

where $R = \lambda_1(1-z)$ and $T = \lambda_2(1-z)$
Multiplying the equation (12) by z^n, summing from $n = 1$ to ∞ and adding with equation (11), we get

$$R_1(0,z) = P_0(0)B^*(\lambda_1) \tag{24}$$

Multiplying equation (13) by z^n, summing from $n = 0$ to ∞ , we get

$$R_i(0,z) = \prod_{l=1}^{i-1}\theta_l V_l^*(T)B^*(\lambda_1)P_0(0), i = 2,3,...,K \tag{25}$$

Put $n = 0$ in equation (13),we get

$$R_{i,0}(0) = \prod_{l=1}^{i-1}\theta_l V_l^*(\lambda_2)B^*(\lambda_1)P_0(0), i = 2,3,...,K \tag{26}$$

From equation (23)

$$\left.\begin{aligned} P(0,z) &= \lambda_0 zQ + (1-\theta_1)V_1^*(T)P_0(0)B^*(\lambda_1) \\ &\quad + \sum_{i=2}^{K}(1-\theta_i)V_i^*(T)\prod_{l=1}^{i-1}\theta_l V_l^*(T)B^*(\lambda_1)P_0(0) \\ &\quad - (1-\theta_1)V_1^*(\lambda_2)B^*(\lambda_1)P_0(0) - P_0(0)B^*(\lambda_1) \\ &\quad - \sum_{i=2}^{K}(1-\theta_i)V_i^*(\lambda_2)\prod_{l=1}^{i-1}\theta_l V_l^*(\lambda_2)B^*(\lambda_1)P_0(0) \end{aligned}\right\} \tag{27}$$

From equation (8),

$$\lambda_0 Q = (1-\theta_1)V_1^*(\lambda_2)B^*(\lambda_1)P_0(0)+\sum_{i=2}^{K}(1-\theta_i)V_i^*(\lambda_2)\prod_{l=1}^{i-1}\theta_l V_l^*(\lambda_2)B^*(\lambda_1)P_0(0) \quad (28)$$

Substituting equation(28) in (27), we get

$$P(0,z) = \frac{[(z-1)A+(1-\theta_1)V_1^*(T)+A_1-1]}{[z-B^*(R)]}B^*(\lambda_1)P_0(0) \quad (29)$$

where

$$A = (1-\theta_1)V_1^*(\lambda_2)+\sum_{i=2}^{K}(1-\theta_i)V_i^*(\lambda_2)\prod_{l=1}^{i-1}\theta_l V_l^*(\lambda_2)$$
$$A_1 = \sum_{i=2}^{K}(1-\theta_i)V_i^*(T)\prod_{l=1}^{i-1}\theta_l V_l^*(T)$$

Now

$$\left.\begin{aligned} P(z) &= \int_0^\infty P(x,z)dx = P(0,z)\frac{[1-B^*(R)]}{R} \\ \text{and } \ R_i(z) &= \int_0^\infty R_i(x,z)dx = R_i(0,z)\frac{[1-V_i^*(T)]}{T}, i=1,2,...,K. \end{aligned}\right\} \quad (30)$$

To find the unknown probability $P_0(0)$ we use the normalization condition

$$Q+P(1)+\sum_{i=1}^{K}R_i(1)=1,$$

we get

$$P_0(0) = \frac{\lambda_0(1+\lambda_1 B^{*'}(0))}{B^*(\lambda_1)C_1} \quad (31)$$

where

$$C_1 = [1+(\lambda_1-\lambda_0)B^{*'}(0)]A-\lambda_0[1+(\lambda_1-\lambda_2)B^{*'}(0)]\Big[V_1^{*'}(0)+\sum_{i=2}^{K}V_i^{*'}(0)\prod_{l=1}^{i-1}\theta_l\Big]$$

$P(z)$ and $R_i(z)$ are the probability generating functions of number of customers in the queue when the server is busy and the server is on the i^{th} phase of vacation.

Substituting equation (31) in (28),we get

$$Q = \frac{(1+\lambda_1 B^{*'}(0))A}{C_1} \quad (32)$$

Equations in (30), together with (24), (25), (28), (29), (31) and (32) gives the probability generating function of number of customers in the queue with server is busy and the server is on the i^{th} phase of vacation ($i=1,2,...,K$)respectively.

3. Some Operating Characteristics

In this section we derive the operating characteristics mean and variance number of customers in the queue when the server is busy and mean and variance number of customers in the queue when the server is on the $i^{th}(i = 1, 2, ..., K)$ phase of vacation.

(i) Mean number of customers in the queue when the server is busy:

$$L_b = \frac{\lambda_0[\lambda_1 B^{*''}(0)C_2 - B^{*'}(0)(1 + \lambda_1 B^{*'}(0))C_3]}{2(1 + \lambda_1 B^{*'}(0))C_1}$$

(ii) Variance of number of customers in the queue when the server is busy:

$$V_b = \frac{\lambda_0[2C_1C_8 - 3\lambda_0 C_9]}{12(1 + \lambda_1 B^{*'}(0))^2 C_1^2}$$

(iii) Mean number of customers in the queue when the server is on vacation:

$$L_v = \frac{\lambda_0\lambda_2(1 + \lambda_1 B^{*'}(0))C_{10}}{2C_1}$$

(iv) Variance of number of customers in the queue when the server is on vacation:

$$V_v = \frac{\lambda_0\lambda_2(1 + \lambda_1 B^{*'}(0))[6C_1C_{10} - 4\lambda_2 C_1 C_{11} - 3\lambda_0\lambda_2(1 + \lambda_1 B^{*'}(0))C_{10}^2]}{12C_1^2}$$

where

$$C_2 = A - \lambda_2(1 - \theta_1)V_1^{*'}(0) + C_5$$

$$C_3 = \lambda_2^2(1 - \theta_1)V_1^{*''}(0) + C_6$$

$$C_4 = -\lambda_2^3(1 - \theta_1)V_1^{*'''}(0) + C_7$$

$$C_5 = -\lambda_2\Big[\theta_1 V_1^{*'}(0) + \sum_{i=2}^{K} V_i^{*'}(0)\prod_{l=1}^{i-1}\theta_l\Big]$$

$$C_6 = \lambda_2^2\Big[\theta_1 V_1^{*''}(0) + \sum_{i=2}^{K} V_i^{*''}(0)\prod_{l=1}^{i-1}\theta_l + 2\sum_{i=1}^{K-1} V_i^{*'}(0)\sum_{n=i+1}^{K} V_n^{*'}(0)\prod_{l=1}^{n-1}\theta_l\Big]$$

$$\begin{aligned} C_7 = -\lambda_2^3\Big[&\theta_1 V_1^{*'''}(0) + \sum_{i=2}^{K} V_i^{*'''}(0)\prod_{l=1}^{i-1}\theta_l \\ &+ 3\Big[\sum_{j=1}^{K-1} V_i^{*''}(0)\sum_{n=j+1}^{K} V_n^{*'}(0)\prod_{l=1}^{n-1}\theta_l + \sum_{j=1}^{K-1} V_j^{*'}(0)\sum_{n=j+1}^{K} V_n^{*''}(0)\prod_{l=1}^{n-1}\theta_l\Big] \\ &+ 6\sum_{m=1}^{K-2} V_m^{*'}(0)\sum_{j=m+1}^{K-1} V_j^{*'}(0)\sum_{n=j+1}^{K} V_n^{*'}(0)\prod_{l=1}^{n-1}\theta_l\Big] \end{aligned}$$

$$\begin{aligned} C_8 = &\lambda_1^2[3\lambda_1 B^{*''^2}(0) - 2B^{*'''}(0)(1 + \lambda_1 B^{*'}(0))]C_2 + 3\lambda_1 B^{*''}(0)(1 + \lambda_1 B^{*'}(0)) \\ &\times [C_2 + C_3] - B^{*'}(0)(1 + \lambda_1 B^{*'}(0))^2[2C_4 + 3C_3] \end{aligned}$$

$$\begin{aligned} C_9 = &\lambda_1^2 B^{*''^2}(0)C_2^2 + B^{*'^2}(0)(1 + \lambda_1 B^{*'}(0))^2 C_3^2 - 2\lambda_1 B^{*'}(0)B^{*''}(0) \\ &\times (1 + \lambda_1 B^{*'}(0))C_2C_3 \end{aligned}$$

$$C_{10} = V_1^{*''}(0) + \sum_{i=2}^{K} V_i^{*''}(0)\prod_{l=1}^{i-1}\theta_l + 2\sum_{i=1}^{K-1} V_i^{*'}(0)\sum_{n=i+1}^{K} V_n^{*'}(0)\prod_{l=1}^{n-1}\theta_l$$

$$C_{11} = V_1^{*'''}(0) + \sum_{i=2}^{K} V_i^{*'''}(0) \prod_{l=1}^{i-1} \theta_l + 3\Big[\sum_{j=1}^{K-1} V_i^{*''}(0) \sum_{n=j+1}^{K} V_n^{*'}(0) \prod_{l=1}^{n-1} \theta_l + \sum_{j=1}^{K-1} V_j^{*'}(0)$$
$$\times \sum_{n=j+1}^{K} V_n^{*''}(0) \prod_{l=1}^{n-1} \theta_l\Big] + 6\sum_{m=1}^{K-2} V_m^{*'}(0) \sum_{j=m+1}^{K-1} V_j^{*'}(0) \sum_{n=j+1}^{K} V_n^{*'}(0) \prod_{l=1}^{n-1} \theta_l$$

4. Some Particular Cases

In this section, we present six particular cases by assuming particular form to the parameters and/or particular probability distribution to service time and/or vacation time.

Case 1: Now we take $\lambda_0 = \lambda_1 = \lambda_2 = \lambda$ and $\theta_i = 0(M/G/1$ Queue with single vacation)

$$Q = \frac{(1 + \lambda B^{*'}(0))V_1^*(\lambda)}{[V_1^*(\lambda) - \lambda V_1^{*'}(0)]}$$
$$L_b = \frac{\lambda^2[B^{*''}(0)[V_1^*(\lambda) - \lambda V_1^{*'}(0)] - \lambda B^{*'}(0)V_1^{*''}(0)(1 + \lambda B^{*'}(0))]}{2(1 + \lambda B^{*'}(0))[V_1^*(\lambda) - \lambda V_1^{*'}(0)]}$$
$$L_v = \frac{\lambda^2 V_1^{*''}(0)(1 + \lambda B^{*'}(0))}{2[V_1^*(\lambda) - \lambda V_1^{*'}(0)]}$$

The results concide with result of $M/G/1$ queue with single vacation (page 21, [16]).

Case 2: We take $\lambda_0 = \lambda_1 = \lambda_2 = \lambda$

$$Q = \frac{(1 + \lambda B^{*'}(0))F_1}{F_2}$$
$$L_b = \frac{\lambda[\lambda B^{*''}(0)F_2 - B^{*'}(0)(1 + \lambda B^{*'}(0))F_3]}{2(1 + \lambda B^{*'}(0))F_2}$$
$$V_b = \frac{\lambda[2F_1F_5 - 3\lambda F_6]}{12(1 + \lambda B^{*'}(0))^2 F_2^2}$$
$$L_v = \frac{\lambda^2(1 + \lambda B^{*'}(0))C_{10}}{2F_2}$$
$$V_v = \frac{\lambda^2(1 + \lambda B^{*'}(0))[6F_2C_{10} - 4\lambda F_2 C_{11} - 3\lambda^2(1 + \lambda B^{*'}(0))C_{10}^2]}{12F_2^2}$$

where

$$F_1 = (1 - \theta_1)V_1^*(\lambda) + \sum_{i=2}^{k}(1 - \theta_i)V_i^*(\lambda) \prod_{l=1}^{i-1} \theta_l V_l^*(\lambda)$$
$$F_2 = F_1 - \lambda\Big[V_1^{*'}(0) + \sum_{i=2}^{K} V_i^{*'}(0) \prod_{l=1}^{i-1} \theta_l\Big]$$

$$F_3 = \lambda^2\Big[V_1^{*''}(0) + \sum_{i=2}^{K} V_i^{*''}(0)\prod_{l=1}^{i-1}\theta_l + 2\sum_{i=1}^{K-1} V_i^{*'}(0)\sum_{n=i+1}^{K} V_n^{*'}(0)\prod_{l=1}^{n-1}\theta_l\Big]$$

$$F_4 = -\lambda^3\Big[V_1^{*'''}(0) + \sum_{i=2}^{K} V_i^{*'''}(0)\prod_{l=1}^{i-1}\theta_l + 3\Big[\sum_{j=1}^{K-1} V_j^{*''}(0)\sum_{n=j+1}^{K} V_n^{*'}(0)\prod_{l=1}^{n-1}\theta_l$$

$$+\sum_{j=1}^{K-1} V_j^{*'}(0)\sum_{n=j+1}^{K} V_n^{*''}(0)\prod_{l=1}^{n-1}\theta_l\Big] + 6\sum_{m=1}^{K-2} V_m^{*'}(0)\sum_{j=m+1}^{K-1} V_j^{*'}(0)\sum_{n=j+1}^{K} V_n^{*'}(0)\prod_{l=1}^{n-1}\theta_l\Big]$$

$$F_5 = \lambda^2[3\lambda B^{*''^2}(0) - 2B^{*'''}(0)(1+\lambda B^{*'}(0))]F_2 + 3\lambda B^{*''}(0)(1+\lambda B^{*'}(0))[F_2+F_3]$$

$$- B^{*'}(0)(1+\lambda B^{*'}(0))^2[2F_4+3F_3]$$

$$F_6 = \lambda^2 B^{*''^2}(0)F_2^2 + B^{*'^2}(0)(1+\lambda B^{*'}(0))^2F_3^2 - 2\lambda B^{*'}(0)B^{*''}(0)(1+\lambda B^{*'}(0))F_2F_3$$

Case 3: We take $\theta_i = 0$

$$Q = \frac{(1+\lambda_1 B^{*'}(0))V_1^*(\lambda_2)}{F_7}$$

$$L_b = \frac{\lambda_0[\lambda_1 B^{*''}(0)[V_1^*(\lambda_2) - \lambda_2 V_1^{*'}(0)] - \lambda_2^2 B^{*'}(0)V_1^{*''}(0)(1+\lambda_1 B^{*'}(0))]}{2(1+\lambda_1 B^{*'}(0))F_7}$$

$$V_b = \frac{\lambda_0[2F_7F_8 - 3\lambda_0 F_9]}{12(1+\lambda_1 B^{*'}(0))^2 F_7^2}$$

$$L_v = \frac{\lambda_0\lambda_2(1+\lambda_1 B^{*'}(0))V_1^{*''}(0)}{2F_7}$$

$$V_v = \frac{1}{12F_7^2}\Big\{\lambda_0\lambda_2(1+\lambda_1 B^{*'}(0))[6V_1^{*''}(0)F_7 - 4\lambda_2 V_1^{*'''}(0)F_7$$

$$- 3\lambda_0\lambda_2(1+\lambda_1 B^{*'}(0))V_1^{*''^2}(0)]\Big\}$$

where

$$F_7 = [1+(\lambda_1-\lambda_0)B^{*'}(0)]V_1^*(\lambda_2) - \lambda_0[1+(\lambda_1-\lambda_2)B^{*'}(0)]V_1^{*'}(0)$$

$$F_8 = \lambda_1^2[3\lambda_1 B^{*''^2}(0) - 2B^{*'''}(0)(1+\lambda_1 B^{*'}(0))][V_1^*(\lambda_2) - \lambda_2 V_1^{*'}(0)$$

$$+ 3\lambda_1 B^{*''}(0)[1+\lambda_1 B^{*'}(0)][[V_1^*(\lambda_2) - \lambda_2 V_1^{*'}(0)] + \lambda_2^2 V_1^{*''}(0)]$$

$$- \lambda_2^2 B^{*'}(0)[1+\lambda_1 B^{*'}(0)]^2[3V_1^{*''}(0) - 2\lambda_2 V_1^{*'''}(0)]]$$

$$F_9 = \lambda_1^2 B^{*''^2}(0)[V_1^*(\lambda_2) - \lambda_2 V_1^{*'}(0)]^2 + \lambda_2^4 B^{*'^2}(0)[1+\lambda_1 B^{*'}(0)]^2 V_1^{*''^2}(0)$$

$$- 2\lambda_1\lambda_2^2 B^{*'}(0)B^{*''}(0)[1+\lambda_1 B^{*'}(0)][V_1^*(\lambda_2) - \lambda_2 V_1^{*'}(0)]V_1^{*''}(0)$$

Case 4: The Service time and Vacation time follows exponential distribution i.e., $B(x) = 1 - e^{\mu x}, B^*(s) = \dfrac{\mu}{s+\mu}, B^{*'}(0) = \dfrac{-1}{\mu}, B^{*''}(0) = \dfrac{2}{\mu^2}, B^{*'''}(0) = \dfrac{-6}{\mu^3}, V_i(x) = 1 - e^{-\eta_i(x)}, V_i^*(s) = \dfrac{\eta_i}{s+\eta_i}, V_i^{*'}(0) = \dfrac{-1}{\nu_i}, V_i^{*''}(0) = \dfrac{2}{\nu_i^2}, V_i^{*'''}(0) = \dfrac{-6}{\nu_i^3}, i = 1, 2, ..., K$

$$Q = \frac{(\mu-\lambda_1)F_{10}}{F_{11}}$$

$$L_b = \frac{\lambda_0[\lambda_1 F_{13} + \lambda_2^2(\mu-\lambda_1)F_{12}]}{(\mu-\lambda_1)F_{11}}$$

$$V_v = \frac{\lambda_0\lambda_2(\mu-\lambda_1)[2\lambda_2 F_{11}F_{14} + F_{11}F_{12} - \lambda_0\lambda_2(\mu-\lambda_1)F_{12}^2]}{F_{11}^2}$$

where

$$F_{10} = (1-\theta_1)\frac{\eta_1}{\eta_1+\lambda_2} + \sum_{i=2}^{K}(1-\theta_i)\frac{\eta_i}{\lambda_2+\eta_i}\prod_{l=1}^{i-1}\theta_l\frac{\eta_l}{\lambda_2+\eta_l}$$

$$F_{11} = [\mu + (\lambda_0-\lambda_1)]F_{10} + \lambda_0[\mu+(\lambda_2-\lambda_1)]\Big[\frac{1}{\nu_1} + \sum_{i=2}^{K}\frac{1}{\nu_i}\prod_{l=1}^{i-1}\theta_l\Big]$$

$$F_{12} = \frac{1}{\nu_1^2} + \sum_{i=2}^{K}\frac{1}{\nu_i^2}\prod_{l=1}^{i-1}\theta_l + \sum_{i=1}^{K-1}\frac{1}{\nu_i}\sum_{n=i+1}^{K}\frac{1}{\nu_n}\prod_{l=1}^{n-1}\theta_l$$

$$F_{13} = F_{10} + \lambda_2\Big[\frac{1}{\nu_1} + \sum_{i=2}^{K}\frac{1}{\nu_i}\prod_{l=1}^{i-1}\theta_l\Big]$$

$$F_{14} = \frac{1}{\nu_1^3} + \sum_{i=2}^{K}\frac{1}{\nu_i^3}\prod_{l=1}^{i-1}\theta_l + \Big[\sum_{j=1}^{K-1}\frac{1}{\nu_j^2}\sum_{n=j+1}^{K}\frac{1}{\nu_n}\prod_{l=1}^{n-1}\theta_l + \sum_{j=1}^{K-1}\frac{1}{\nu_j}\sum_{n=j+1}^{K}\frac{1}{\nu_n^2}\prod_{l=1}^{n-1}\theta_l\Big]$$
$$+ \sum_{m=1}^{K-2}\frac{1}{\nu_m}\sum_{j=m+1}^{K-1}\frac{1}{\nu_j}\sum_{n=j+1}^{K}\frac{1}{\nu_n}\prod_{l=1}^{n-1}\theta_l$$

$$F_{15} = 2\lambda_1^2F_{13} + \lambda_1(\mu-\lambda_1)[F_{13} + 2\lambda_2^2F_{12}] + \lambda_2^2(\mu-\lambda_1)^2[F_{12} - 2\lambda_2F_{14}]$$

$$F_{16} = \lambda_1^2F_{13}^2 + \lambda_2^4(\mu-\lambda_1)^2F_{12}^2 + 2\lambda_1\lambda_2^2(\mu-\lambda_1)F_{12}F_{13}$$

Case 5: The Service time follows exponential distribution
i.e., $B(x) = 1 - e^{\mu x}, B^*(s) = \frac{\mu}{s+\mu}, B^{*'}(0) = \frac{-1}{\mu}, B^{*''}(0) = \frac{2}{\mu^2}, B^{*'''}(0) = \frac{-6}{\mu^3}$

$$Q = \frac{(\mu-\lambda_1)A}{F_{17}}$$

$$L_b = \frac{\lambda_0[2\lambda_1C_2 + (\mu-\lambda_1)C_3]}{2(\mu-\lambda_1)F_{17}}$$

$$V_b = \frac{\lambda_0[2F_{17}F_{18} - 3\lambda_0F_{19}]}{12(\mu-\lambda_1)^2F_{17}^2}$$

$$L_v = \frac{\lambda_0\lambda_2(\mu-\lambda_1)C_{10}}{2F_{17}}$$

$$V_v = \frac{\lambda_0\lambda_2(\mu-\lambda_1)[6F_{17}C_{10} - 4\lambda_2F_{17}C_{11} - 3\lambda_0\lambda_2(\mu-\lambda_1)C_{10}^2]}{12F_{17}^2}$$

where

$$F_{17} = [\mu + (\lambda_0-\lambda_1)]A - \lambda_0[\mu+(\lambda_2-\lambda_1)]\Big[V_1^{*'}(0) + \sum_{i=2}^{K}V_i^{*'}(0)\prod_{l=1}^{i-1}\theta_l\Big]$$

$$F_{18} = 12\lambda_1^2C_2 + 6\lambda_1(\mu-\lambda_1)[C_2+C_3] + (\mu-\lambda_1)^2[2C_4 + 3C_3]$$

$$F_{19} = 4\lambda_1^2 C_2^2 + (\mu-\lambda_1)^2 C_3^2 + 4\lambda_1(\mu-\lambda_1)C_2C_3$$

Case 6: The Vacation time follows exponential distribution
i.e., $V_i(x) = 1 - e^{-\eta_i(x)}, V_i^*(s) = \frac{\eta_i}{s+\eta_i}, V_i^{*'}(0) = \frac{-1}{\nu_i}, V_i^{*''}(0) = \frac{2}{\nu_i^2},$
$V_i^{*'''}(0) = \frac{-6}{\nu_i^3}, i = 1, 2, ..., K.$

$$Q = \frac{(1+\lambda_1 B^{*'}(0))F_{10}}{F_{20}}$$

$$L_b = \frac{\lambda_0[\lambda_1 B^{*''}(0)F_{13} - 2\lambda_2^2 B^{*'}(0)(1+\lambda_1 B^{*'}(0))F_{12}]}{2(1+\lambda_1 B^{*'}(0))F_{20}}$$

$$V_b = \frac{\lambda_0[2F_{20}F_{21} - 3\lambda_0 F_{22}]}{12(1+\lambda_1 B^{*'}(0))^2 F_{20}^2}$$

$$L_v = \frac{\lambda_0\lambda_2(1+\lambda_1 B^{*'}(0))F_{12}}{F_{20}}$$

$$V_v = \frac{\lambda_0\lambda_2(1+\lambda_1 B^{*'}(0))[2\lambda_2 F_{14}F_{20} + F_{12}F_{20} - \lambda_0\lambda_2(1+\lambda_1 B^{*'}(0))F_{12}^2]}{F_{20}^2}$$

where

$$F_{20} = [1+(\lambda_1-\lambda_0)B^{*'}(0)]F_{10} + \lambda_0[1+(\lambda_1-\lambda_2)B^{*'}(0)]\Big[\frac{1}{\nu_1} + \sum_{i=2}^{K}\frac{1}{\nu_i}\prod_{l=1}^{i-1}\theta_l\Big]$$

$$F_{21} = \lambda_1^2[3\lambda_1 B^{*''^2}(0) - 2B^{*'''}(0)(1+\lambda_1 B^{*'}(0))]F_{13} + 3\lambda_1 B^{*''}(0)(1+\lambda_1 B^{*'}(0))$$
$$\times [F_{13} + 2\lambda_2^2 F_{12}] + 6\lambda_2^2 B^{*'}(0)(1+\lambda_1 B^{*'}(0))^2[2\lambda_2 F_{14} - F_{12}]$$

$$F_{22} = \lambda_1^2 B^{*''^2}(0)F_{13}^2 + 4\lambda_2^4 B^{*'^2}(0)(1+\lambda_1 B^{*'}(0))^2 F_{12}^2$$
$$- 4\lambda_1\lambda_2^2 B^{*'}(0)B^{*''}(0)(1+\lambda_1 B^{*'}(0))F_{12}F_{13}$$

5. Numerical Results

In this section, We present some numerical results in order to illustrate the effect of various parameters on the performance measures of the models in section 4. The effect of the parameters arrival rate, service rate, vacation rate and the number of phases of vacation on the system performance measures (i) the mean number of customers when the server is busy(L_b), (ii) the mean number of customers in the queue when the server is on vacation(L_v), (iii) the variance of the number of customers in the queue when the server is busy(V_b) and (iv) the variance of the number of customers in the queue when the server is on vacation(V_v) have been numerically analysed. Figures $1((a),(b),(c))$ represents the graph of mean number of customers when $K = 3, 5$and 7 by varying the service rate. Tables $1-3$ shows the variance of number of customers. In all the figures, it is clear that the mean number of customers in the queue when the server is busy is decreasing function with respect to service rate where as its counter part are increasing functions as expected. The variance value with respect to server busy decreases as the service rate increases but in the case of variance with respect to vacation we encounters the contrary concept that is variance increases. For this analysis the values of $\lambda_0 = 0.6$, $\lambda_1 = 0.8$, $\lambda_2 = 0.4$ are fixed.

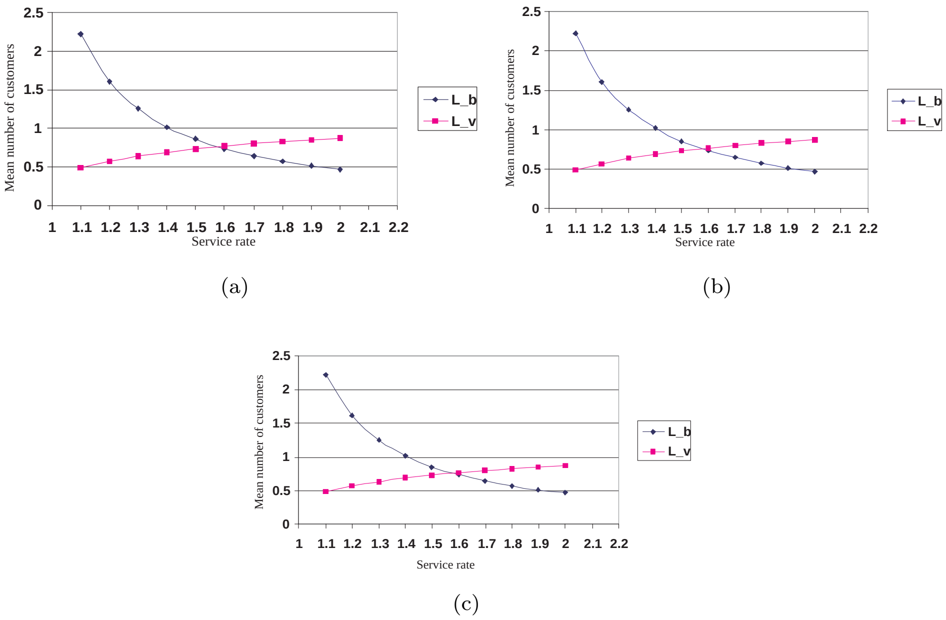

Figure 1. Mean number of customers for $K = 3, 5$ and 7.

Table 1. Variance for $K = 3$.

μ	V_b	V_v
1.1	7.32850	2.07285
1.2	3.86549	2.47989
1.3	2.26881	2.80961
1.4	1.41360	3.08187
1.5	0.90925	3.31033
1.6	0.59128	3.50471
1.7	0.38079	3.67205
1.8	0.23623	3.81760
1.9	0.13409	3.94534
2.0	0.06030	4.05834

Table 2. Variance for $K = 5$.

μ	V_b	V_v
1.1	7.32428	2.07904
1.2	3.86168	2.48731
1.3	2.26535	2.81802
1.4	1.41043	3.09109
1.5	0.90632	3.32024
1.6	0.58856	3.51520
1.7	0.37825	3.68305
1.8	0.23385	3.82904
1.9	0.13185	3.95717
2.0	0.05819	4.07051

6. Conclusion

In the foregoing analysis an $M/G/1$ queue with regular and optional phase vacation and with state dependent arrival rate is considered. For this model the queue length distribution and mean queue length are obtained. An extensive numerical work has been carried out to observe the nature of the operating characteristics.

Table 3. Variance for $K = 7$.

μ	V_b	V_v
1.1	7.32416	2.07920
1.2	3.86157	2.48750
1.3	2.26525	2.81823
1.4	1.41034	3.09132
1.5	0.90624	3.32049
1.6	0.58848	3.51547
1.7	0.37818	3.68333
1.8	0.23379	3.82933
1.9	0.13179	3.95747
2.0	0.05813	4.07081

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