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Numerical Solution of One-Dimensional Heat and Wave Equation by Non-Polynomial Quintic Spline

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Abstract. This paper present a numerical algorithm for the linear one-dimensional heat and wave equation. In this method, a finite difference approach had been used to discrete the time derivative while quintic spline is applied as an interpolation function in the space dimension. We discuss the accuracy of the method by expanding the equation based on Taylor series and minimize the error. The proposed method has eighth-order accuracy in space and fourth-order accuracy in time variables. From the computational point of view, the solution obtained by this method is in excellent agreement with those obtained by previous works and also it is efficient to use. Numerical examples are given to show the applicability and efficiency of the method.

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1. Introduction

Consider one-dimensional heat equation of the form

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \leqslant x \leqslant l, \quad t \geqslant 0, \tag{1}$$

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with initial condition

$$u(x,0) = f_1(x), \qquad 0 \leqslant x \leqslant l, \tag{2}$$

and boundary conditions

$$u(0,t) = p_1(t), \qquad t \geqslant 0,$$

$$u(l,t) = q_1(t), \qquad t \geqslant 0, \tag{3}$$

and one-dimensional wave equation of the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \leqslant x \leqslant l, \quad t \geqslant 0, \tag{4}$$

with initial condition

$$u(x,0) = f_2(x), \qquad \frac{\partial u(x,0)}{\partial t} = f_3(t), \qquad 0 \leqslant x \leqslant l, \tag{5}$$

and boundary conditions

$$u(0,t) = p_2(t), \qquad t \geqslant 0,$$

$$u(l,t) = q_2(t), \qquad t \geqslant 0, \tag{6}$$

where c^2 and l are positive finite real constants and $f_1(x)$, $f_2(x)$, $f_3(x)$, $p_1(t)$, $p_2(t)$, $q_1(t)$ and $q_2(t)$ are real continuous functions.

Some phenomena, which arise in many fields of scientific such as solid state physics, plasma physics, fluid dynamics, mathematical biology and chemical kinetics, can be modeled by partial differential equations. The heat and wave equations are of primary importance in many physical systems such as electro-thermal analogy[1], signal formation[2], draining film [3], water transfer in soils [4], mechanic and physics[5-8], elasticity[9] and etc.

There are several numerical schemes that have been developed for the solution of the heat and wave equation [10-14]. Cubic spline has been used to approximate the one dimensional heat conduction in[15], also the collocation method based on hermite cubic spline applied to solve one-dimensional heat conduction problem in [16].

In this paper, a method based on quintic spline for second-order boundary value problems Eq.(1) and Eq.(4) is presented. This approach will employ consistency relations at mid-knot. In Section 2 the formulation of non-polynomial quintic spline has been developed and the consistency relation obtained is useful to discretize heat equation Eq.(1) and wave equation Eq.(4). In Section 3, we present discretization of the equation by a finite difference approximation to obtain the formulation of proposed method. In Section 4, Truncation error and stability analysis are discussed. In this section we approximate the functions based on Taylor series to minimize the error term and to obtain the class of methods. In Section 5, numerical experiments are conducted to demonstrate the viability and the efficiency of the proposed method computationally.

2. Formulation of Tension Quintic Spline Function

In recent years, many scholars have used non-polynomial spline for solving differential equations [17-19]. The spline function is a piecewise polynomial or non-polynomial of degree n satisfying the continuity of the $(n-1)$th derivative. Tension quintic spline is a non-polynomial function that has six parameters to be determined hence it can satisfy the conditions of two endpoints of the interval and continuity of first, second, third and fourth derivatives. We introduce the set of grid points in the interval $[0, l]$ in space direction

$$x_i = ih \ , \quad h = \frac{l}{n+1} \ , \qquad i = 0, 1, 2, ..., n+1.$$

For each segment, quintic spline $p_i(x)$ is define as

$$p_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 + e_i(e^{\omega(x-x_i)} - e^{-\omega(x-x_i)})$$

$$+ f_i(e^{\omega(x-x_i)} + e^{-\omega(x-x_i)}), \quad i = 0, 1, ..., n, \tag{7}$$

where a_i, b_i, c_i, d_i, e_i and f_i are the unknown coefficients to be determined also ω is free parameter. If $\omega \to 0$ then $p_i(x)$ reduces to quintic spline in the interval $[0, l]$. To derive the unknown coefficients, we define

$$p_i(x_i) = u_i, \ p_i(x_{i+1}) = u_{i+1}, \ p''_i(x_i) = M_i, \ p''_i(x_{i+1}) = M_{i+1},$$

$$p_i^{(4)}(x_i) = S_i, \ p_i^{(4)}(x_{i+1}) = S_{i+1}. \tag{8}$$

From Eq.(7) and Eq.(8), we can determine the unknown coefficients

$$\begin{aligned}
a_i &= u_i - \frac{S_i}{\omega^4} \ , \\
b_i &= \frac{u_{i+1}}{h} - \frac{u_i}{h} + S_i\left(\frac{h}{\omega^4} + \frac{h}{3\omega^2}\right) + S_{i+1}\left(\frac{h}{6\omega^2} - \frac{1}{h\omega^4}\right) - \frac{h}{3}M_i - \frac{h}{6}M_{i+1}, \\
c_i &= \frac{M_i}{2} - \frac{S_i}{2\omega^2} \ , \ d_i = \frac{1}{6h}\left(M_{i+1} - M_i + \frac{S_i}{\omega^2} - \frac{S_{i+1}}{\omega^2}\right) \\
e_i &= \frac{S_{i+1}}{\omega^4\left(e^\theta - e^{-\theta}\right)} - \frac{S_i\left(e^\theta + e^{-\theta}\right)}{2\omega^4\left(e^\theta - e^{-\theta}\right)} \ , \ f_i = \frac{S_i}{2\omega^4},
\end{aligned}$$

where $\theta = \omega h$ and $i = 0, 1, 2, ..., n$.
Finally using the continuity of first derivative at the support points for $i = 2, 3, ..., n-1$, we have

$$\begin{aligned}
&\frac{u_{i+1}}{h} - 2\,\frac{u_i}{h} + \frac{u_{i-1}}{h} - \frac{h}{6}M_{i-1} - \frac{2h}{3}M_i - \frac{h}{6}M_{i+1} = \\
&= S_{i-1}\left(h\omega^{-4} - \frac{h}{6\omega^2} - \frac{\left(e^\theta + e^{-\theta}\right)^2}{2\omega^3\left(e^\theta - e^{-\theta}\right)} + \frac{e^\theta - e^{-\theta}}{2\omega^3}\right),
\end{aligned} \tag{9}$$

and from continuity of third derivative, we have

$$\frac{M_{i+1}}{h} - 2\,\frac{M_i}{h} + \frac{M_{i-1}}{h} = S_{i-1}\left(h\omega^{-2} - \frac{\left(e^{\theta}+e^{-\theta}\right)^2}{2\omega\,\left(e^{\theta}-e^{-\theta}\right)} + \frac{e^{\theta}-e^{-\theta}}{2\omega}\right)$$
$$+S_i\left(-2\,h\omega^{-2} + 2\,\frac{e^{\theta}+e^{-\theta}}{\omega\,\left(e^{\theta}-e^{-\theta}\right)}\right) + S_{i+1}\left(h\omega^{-2} + 2\,\frac{1}{\omega\,\left(e^{\theta}-e^{-\theta}\right)}\right). \qquad (10)$$

From Eq.(9) and Eq.(10), after eliminating S_i we have the following useful relation for $i = 2, 3, ..., n-2$

$$u_{i+2} + 2\,u_{i+1} - 6\,u_i + 2\,u_{i-1} + u_{i-2} =$$
$$= \frac{h^2}{20}\left(\alpha\,M_{i+2} + \beta\,M_{i+1} + \gamma\,M_i + \beta\,M_{i-1} + \alpha\,M_{i-2}\right), \qquad (11)$$

where

$$\alpha = \theta^{-4} - \frac{2}{\theta^3\left(e^{\theta}-e^{-\theta}\right)} - \frac{1}{3\theta\,\left(e^{\theta}-e^{-\theta}\right)},$$
$$\beta = -4\,\theta^{-4} + \frac{4+2\,e^{\theta}+2\,e^{-\theta}}{\theta^3\left(e^{\theta}-e^{-\theta}\right)} + \frac{e^{\theta}+e^{-\theta}-4}{3\theta\,\left(e^{\theta}-e^{-\theta}\right)},$$
$$\gamma = 6\,\theta^{-4} + \frac{4\,e^{\theta}+4\,e^{-\theta}-2}{3\theta\,\left(e^{\theta}-e^{-\theta}\right)} - \frac{4+4\,e^{\theta}+4\,e^{-\theta}}{\theta^3\left(e^{\theta}-e^{-\theta}\right)}.$$

When $\omega \to 0$ so $\theta \to 0$, then $(\alpha, \beta, \gamma) \to (1, 26, 66)$, and the relation defined by Eq.(11) reduce into ordinary quintic spline

$$u_{i+2} + 2\,u_{i+1} - 6\,u_i + 2\,u_{i-1} + u_{i-2} =$$
$$= \frac{h^2}{20}\left(M_{i+2} + 26\,M_{i+1} + 66\,M_i + 26\,M_{i-1} + M_{i-2}\right). \qquad (12)$$

3. Numerical Technique

By using Eq.(12) for $(j+1)$th, (j)th and $(j-1)$th time level we have

$$u_{i+2}^{j+1} + 2\,u_{i+1}^{j+1} - 6\,u_i^{j+1} + 2\,u_{i-1}^{j+1} + u_{i-2}^{j+1} =$$

$$= \frac{h^2}{20}\left(\alpha\,M_{i+2}^{j+1} + \beta\,M_{i+1}^{j+1} + \gamma\,M_i^{j+1} + \beta\,M_{i-1}^{j+1} + \alpha\,M_{i-2}^{j+1}\right), \qquad (13)$$

$$u_{i+2}^{j} + 2\,u_{i+1}^{j} - 6\,u_i^{j} + 2\,u_{i-1}^{j} + u_{i-2}^{j} =$$

$$= \frac{h^2}{20}\left(\alpha\,M_{i+2}^{j} + \beta\,M_{i+1}^{j} + \gamma\,M_i^{j} + \beta\,M_{i-1}^{j} + \alpha\,M_{i-2}^{j}\right), \qquad (14)$$

$$u_{i+2}^{j-1} + 2\,u_{i+1}^{j-1} - 6\,u_i^{j-1} + 2\,u_{i-1}^{j-1} + u_{i-2}^{j-1} =$$

$$= \frac{h^2}{20}\left(\alpha\, M_{i+2}^{j-1} + \beta\, M_{i+1}^{j-1} + \gamma\, M_i^{j-1} + \beta\, M_{i-1}^{j-1} + \alpha\, M_{i-2}^{j-1}\right), \tag{15}$$

that we will use these equations to discretize heat and wave equation.

3.1 *Heat Equation*

We develop an approximation for Eq.(1) in which first order time derivative is replaced by the following finite difference approximation

$$\bar{u}_t^j = \frac{u_i^{j+1} - u_i^{j-1}}{2k} = u_t^j + O\left(k^2\right), \tag{16}$$

and the space derivative is replaced by non-polynomial tension spline approximation

$$\bar{u}_{xx}^j = p''(x_i, t_j) = M_i^j. \tag{17}$$

By using Eq.(16) and Eq.(17) we can develop a new approximation for the solution of Eq.(1), So that the heat equation Eq.(1) is replaced by

$$\eta\, M_i^{j-1} + (1 - 2\,\eta)\, M_i^j + \eta\, M_i^{j+1} = \frac{u_i^{j+1} - u_i^{j-1}}{2c^2 k}, \tag{18}$$

where $0 \leqslant \eta \leqslant 1$ is a free constant.

If Eq.(14) multiplied by $(1 - 2\eta)$ added to Eq.(13) and Eq.(15) multiplied by η and eliminate M_i^j, then we obtain the following relation for heat equation Eq.(1)

$$\begin{aligned}
&\eta\, \left(u\,(i+2,j-1) + u\,(i-2,j-1)\right) + (1-2\,\eta)\,(u\,(i+2,j) + u\,(i-2,j)) \\
&+\eta\,(u\,(i+2,j+1) + u\,(i-2,j+1)) + 2\,\eta\,[u\,(i+1,j-1) \\
&+u\,(i-1,j-1)] + 2\;(1-2\,\eta)\,(u\,(i+1,j) + u\,(i-1,j)) \\
&+2\,\eta\;(u\,(i+1,j+1) + u\,(i-1,j+1)) - 6\,\eta\, u\,(i,j-1) \\
&-6\;(1-2\,\eta)\,u\,(i,j) - 6\,\eta\, u\,(i,j+1) - \frac{h^2}{40c^2k}\,\alpha\,[u\,(i+2,j+1) \\
&-u\,(i+2,j-1) + u\,(i-2,j+1) - u\,(i-2,j-1)] \\
&-\frac{h^2}{40c^2k}\,\beta\,[u\,(i+1,j+1) - u\,(i+1,j-1) + u\,(i-1,j+1) \\
&-u\,(i-1,j-1)] - \frac{h^2}{40c^2k}\,\gamma\;(u\,(i,j+1) - u\,(i,j-1)) = 0, \\
&\qquad\qquad\qquad\qquad j = 1,2,3,... \qquad i = 2,...,N-2.
\end{aligned} \tag{19}$$

3.2 *Wave Equation*

Finite difference approximation for second order time derivative is

$$\bar{u}_{tt}^j = \frac{u_i^{j+1} - 2\,u_i^j + u_i^{j-1}}{k^2} = u_{tt}^j + O\left(k^2\right), \tag{20}$$

as we consider, the space derivative is approximated by the non-polynomial tension spline

$$\bar{u}^j_{xx} = p''(x_i, t_j) = M^j_i. \tag{21}$$

By using Eq.(16) and Eq.(17) for wave equation Eq.(4) we have

$$\eta\, M_i^{j-1} + (1-2\,\eta)\, M_i^j + \eta\, M_i^{j+1} = \frac{u_i^{j+1} - 2\,u_i^j + u_i^{j-1}}{c^2 k^2}, \tag{22}$$

where $0 \leqslant \eta \leqslant 1$ is a free constant.

Again we multiply Eq.(14) by $(1-2\eta)$ and add this to Eq.(13) and Eq.(15) multiplied by η and eliminate M_i^j, then we obtain the following relation for wave equation Eq.(4)

$$\begin{aligned}
&\eta\, \left(u\,(i+2,j-1) + u\,(i-2,j-1)\right) + (1-2\,\eta)\,(u\,(i+2,j) + u\,(i-2,j))\\
&+\eta\,(u\,(i+2,j+1) + u\,(i-2,j+1)) + 2\,\eta\,[u\,(i+1,j-1)\\
&+u\,(i-1,j-1)] + 2\,\,(1-2\,\eta)\,(u\,(i+1,j) + u\,(i-1,j))\\
&+2\,\eta\,\,(u\,(i+1,j+1) + u\,(i-1,j+1))\\
&-6\,\eta\, u\,(i,j-1) - 6\,\,(1-2\,\eta)\, u\,(i,j) - 6\,\eta\, u\,(i,j+1)\\
&-\frac{h^2}{40c^2k}\,\alpha\,[u\,(i+2,j+1) - 2u\,(i+2,j) + u\,(i+2,j-1) + u\,(i-2,j+1)\\
&-2u\,(i-2,j) + u\,(i-2,j-1)] - \frac{h^2}{40c^2k}\,\beta\,[u\,(i+1,j+1) - 2u\,(i+1,j)\\
&+u\,(i+1,j-1) + u\,(i-1,j+1) - 2u\,(i-1,j) + u\,(i-1,j-1)]\\
&-\frac{h^2}{40c^2k}\,\gamma\,\,(u\,(i,j+1) - 2u\,(i,j) + u\,(i,j-1)) = 0,\\
&\qquad\qquad\qquad\qquad j = 1,2,3,... \qquad i = 2,...,N-2.
\end{aligned} \tag{23}$$

4. Error Estimate in Spline Approximation

To estimate the error for heat and wave equation we expand Eq.(19) and Eq.(23) in Taylor series about $u(x_i, t_j)$ and then we find the optimal values for α, β and γ.

4.1 *Error Estimate for Heat Equation*

By expanding Eq.(19) in Taylor series and replace the derivatives involving t by the relation

$$\frac{\partial^{i+j} u}{\partial x^i \partial t^j} = c^{2j}\frac{\partial^{i+2j} u}{\partial x^{i+2j}}, \tag{24}$$

for heat equation Eq.(1), we obtain the local truncation error. The principal part of the local truncation error of the proposed method for heat equation is

$$
\begin{aligned}
T_i^j = & \left(-\frac{\alpha}{10}-\frac{\gamma}{20}-\frac{\beta}{10}+6\right)h^2\left(D_{x,x}\right)(U)(0,0) \\
& +\left(-\frac{\alpha}{5}-\frac{\beta}{20}+\frac{3}{2}\right)h^4\left(D_{x,x,x,x}\right)(U)(0,0) \\
& +\left(6\,\eta\,c^2h^2k^2+\left(-\frac{\alpha}{15}-\frac{\beta}{240}+\frac{11}{60}\right)h^6\right)\left(D_{x,x,x,x,x,x}\right)(U)(0,0) \\
& +\left(-\frac{\alpha}{60}-\frac{\beta}{60}-\frac{\gamma}{120}\right)c^2h^2k^2\left(D_{x,x,x,x,x,x}\right)(U)(0,0) \\
& +\left(\left(-\frac{\alpha}{30}-\frac{\beta}{120}+\frac{3}{2}\eta\right)c^2h^4k^2\right)\left(D_{x,x,x,x,x,x}\right)(U)(0,0) \\
& +[\left(-\frac{\alpha}{30}-\frac{\beta}{120}+\frac{3}{2}\eta\right)h^4k^2c^2+ \\
& +\left(-\frac{2}{225}\,\alpha-\frac{1}{7200}\,\beta+\frac{43}{3360}\right)h^8]\left(D_{x,x,x,x,x,x,x,x}\right)(U)(0,0) \\
& + \quad ..... \quad . \qquad (25)
\end{aligned}
$$

By choosing suitable values of parameters α, β, γ and η we obtain various classes of the proposed method.

Remark1. If $\theta \to 0$ in Eq.(11) we have $\alpha = 1$, $\beta = 26$ and $\gamma = 66$ which is ordinary quintic spline.

Remark2. If we choose $\alpha = \frac{7}{6}$, $\beta = \frac{76}{3}$ and $\gamma = 67$ in Eq.(25) we obtain a new scheme of order $O(h^8 + h^4k^4)$, furthermore by choosing $\eta = \frac{1}{6}$ we can optimize our scheme, too.

4.2 *Error Estimate for Wave Equation*

For wave equation, we expand Eq.(23) in Taylor series and replace the derivatives involving t by the relation

$$\frac{\partial^{i+j}u}{\partial x^i \partial t^j} = c^j \frac{\partial^{i+j}u}{\partial x^{i+j}}, \qquad (26)$$

and then we drive the local truncation error. The principal part of the local truncation error of the proposed method for wave equation is

$$
\begin{aligned}
T_i^j = & \left(-\frac{\alpha}{10}-\frac{\gamma}{20}-\frac{\beta}{10}+6\right) h^2\left(D_{x,x}\right)(U)(0,0) \\
& +[\left(-\frac{\alpha}{5}-\frac{\beta}{20}+\frac{3}{2}\right) h^4+ \\
& +\left(-\frac{\alpha}{120}-\frac{\gamma}{240}-\frac{\beta}{120}+6\eta\right) h^2k^2c]\left(D_{x,x,x,x}\right)(U)(0,0) \\
& +[\left(-\frac{\alpha}{15}-\frac{\beta}{240}+\frac{11}{60}\right) h^6+ \\
& +\left(-\frac{\alpha}{60}-\frac{\beta}{240}+\frac{3}{2}\eta\right) h^4k^2c]\left(D_{x,x,x,x,x,x}\right)(U)(0,0) \\
& +\left(\left(-\frac{\alpha}{3600}-\frac{\gamma}{7200}-\frac{\beta}{3600}+\frac{1}{2}\eta\right) h^2k^4c^2\right)\left(D_{x,x,x,x,x,x}\right)(U)(0,0) \\
& +\left(\left(-\frac{2}{225}\alpha-\frac{1}{7200}\beta+\frac{43}{3360}\right) h^8\right)\left(D_{x,x,x,x,x,x,x,x}\right)(U)(0,0) \\
& + \; ..... \; .
\end{aligned} \tag{27}
$$

Remark3. If we choose $\alpha = \frac{7}{6}$, $\beta = \frac{76}{3}$ and $\gamma = 67$ in Eq.(27) we obtain a new scheme of order $O(h^8 + h^4k^4)$, furthermore by choosing $\eta = \frac{1}{12}$ we can optimize our scheme, too.

5. Stability Analysis

In this section, we discuss stability of the proposed method for numerical solution of heat and wave equation. we assume that the solution of Eq.(19) and Eq.(23) at grid point (lh, jk) is

$$u_l^j = \xi^j e^{li\theta}, \tag{28}$$

where $i = \sqrt{-1}$, θ is a real number and ξ is a complex number.
By substituting Eq.(28) in Eq.(19) and Eq.(23), we obtain a quadratic equation as follow

$$Q\xi^2 + \phi\xi + \psi = 0. \tag{29}$$

For heat equation we have

$$Q = \cos(2\theta)\left(2\eta + \frac{h^2\alpha}{20ck}\right) + \cos(\theta)\left(4\eta + \frac{h^2\beta}{20ck}\right) - 6\eta + \frac{h^2\gamma}{20ck},$$

$$\phi = \cos(2\theta)(1 - 2\eta) + \cos(\theta)(2 - 4\eta) - 6 + 12\eta,$$

$$\psi = \cos(2\theta)\left(2\eta - \frac{h^2\alpha}{20ck}\right) + \cos(\theta)\left(4\eta - \frac{h^2\beta}{20ck}\right) - 6\eta - \frac{h^2\gamma}{20ck}.$$

By using RouthHurwitz criteria and using transformation $\xi = \frac{1+z}{1-z}$ in Eq.(29), we have

$$(Q - \phi + \psi)\xi^2 + 2(Q - \psi)\xi + (Q + \phi + \psi) = 0. \tag{30}$$

If $|\xi| < 1$, then the difference scheme Eq.(19) is stable. It is sufficient to show that $Q - \phi + \psi > 0$, $2(Q - \psi) > 0$ and $Q + \phi + \psi > 0$.
From the above relation we have
$(i) Q - \phi + \psi = \cos^2(\theta)(12\eta - 2) + \cos(\theta)(12\eta - 2) - 30\eta + 7$,
$(ii) Q - \psi = \frac{h^2}{10ck}(\alpha cos(2\theta) + \beta cos(\theta) + \gamma)$,
$(iii) Q + \phi + \psi = \cos^2(\theta)(4\eta + 2) + \cos(\theta)(4\eta + 2) - 2\eta - 7$.
If $\eta > \frac{1}{6}$ and $\eta > \frac{30}{132}$ then $Q - \phi + \psi > 0$. For $\alpha = \frac{7}{6}$, $\beta = \frac{76}{3}$ and $\gamma = 67$ we have $Q - \psi > 0$ and if $\eta > -\frac{1}{2}$ and $\eta > -\frac{30}{12}$ then $Q + \phi + \psi > 0$.
Thus our method is stable for heat equation.

For wave equation we have

$$Q = \cos(2\theta)\left(2\eta - \frac{h^2\alpha}{10ck^2}\right) + \cos(\theta)\left(4\eta - \frac{h^2\beta}{10ck^2}\right) - 6\eta - \frac{h^2\gamma}{20ck^2},$$

$$\phi = \cos(2\theta)\left(2 - 4\eta + \frac{h^2\alpha}{5ck^2}\right) + \cos(\theta)\left(4 - 8\eta + \frac{h^2\beta}{5ck^2}\right) - 6 + 12\eta + \frac{h^2\gamma}{10ck^2},$$

$$\psi = \cos(2\theta)\left(2\eta - \frac{h^2\alpha}{10ck^2}\right) + \cos(\theta)\left(4\eta - \frac{h^2\beta}{10ck^2}\right) - 6\eta - \frac{h^2\gamma}{20ck^2},$$

thus we have $(Q - \phi + \psi)\xi^2 + (Q + \phi + \psi) = 0$. In order to $|\xi| < 1$, we must have $\phi < 0$ and $Q + \psi > 0$.
Obviously we have $Q + \psi > 0$ for each η, if $\eta > \frac{1}{2} + \frac{7h^2}{120ck^2}$ then $\phi < 0$, therefore our scheme will be stable for wave equation.

6. Numerical Example

We applied the presented method to the following heat and wave equations. For this purpose, we consider three examples for heat equations and two examples for wave equation.
We applied proposed method with $(\alpha, \beta, \gamma) = (1, 26, 66)$ (method I) which is the ordinary quintic spline of with order $O(h^6 + k^4)$ and if we select $(\alpha, \beta, \gamma) = (\frac{7}{6}, \frac{76}{3}, 67)$ we obtain a new method which is of order $O(h^8 + h^4k^4)$ (method II).

Example1:
We consider Eq.(1) with $c = \frac{1}{\pi}$, $f_1(x) = sin(\pi x)$ and $p_1(t) = q_1(t) = 0$. The exact solution for this problem is

$$u(x,t) = e^{-t}\sin(\pi x).$$

Table 1. $Absolute\ error for Example.1$

x_i	t_j	Method I	Method II	Method in [13]
0.125	0.01	4.1×10^{-7}	3×10^{-7}	1×10^{-4}
0.125	0.05	8×10^{-7}	7×10^{-7}	2.5×10^{-4}
0.25	0.01	5.3×10^{-7}	4.9×10^{-7}	4×10^{-4}
0.25	0.05	1×10^{-6}	9.5×10^{-7}	4.2×10^{-4}
0.5	0.01	5×10^{-7}	4.9×10^{-7}	3×10^{-4}
0.5	0.05	1×10^{-6}	9.6×10^{-7}	2×10^{-4}

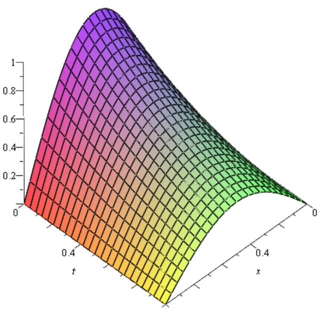

Figure 1. Space-Time graph of the solution up to t=1 s, with $c = \frac{1}{\pi}$, $t = 0.01$ and $h = 0.001$ for example 1.

This problem is solved by different values of the step size in the x-direction h and time step size $\Delta t = 0.01$. The computed solutions by proposed method are compared with the exact solution at the grid points and the maximum absolute errors are tabulated in Table 1. Also the results are compared with the solutions obtained in [13]. The space−time graph of the estimated solution is given in Figs. 1. The maximum absolute error of this example by method II is 3.6×10^{-9} and by method I is 3×10^{-6}.

Example2:

We consider Eq.(1) with $c = 1$, $f_1(x) = cos(\frac{\pi}{2}x)$ and $p_1(t) = e^{-\frac{\pi^2}{4}t}$ and $q_1(t) = 0$. The exact solution for this problem is

$$u(x,t) = e^{-\frac{\pi^2}{4}t} \cos\left(\frac{\pi}{2}x\right).$$

Table 2. *Absolute error for Example.2*

x_i	t_j	$Method I$	$Method II$
0.1	0.0003	6.5×10^{-9}	7×10^{-10}
0.1	0.0005	2.5×10^{-9}	4×10^{-10}
0.15	0.0003	2.3×10^{-9}	3×10^{-10}
0.15	0.0005	2.5×10^{-9}	1.6×10^{-10}
0.2	0.0003	2.2×10^{-9}	1.4×10^{-10}
0.2	0.0005	2.1×10^{-9}	1×10^{-10}

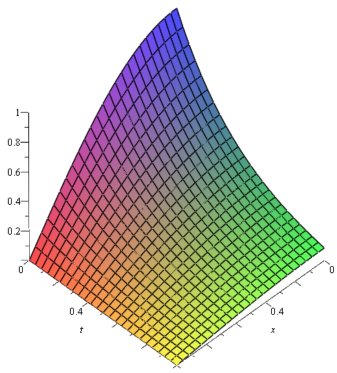

Figure 2. Space-Time graph of the solution up to t=1 s, with $c = 1$, $t = 0.0001$ and $h = 0.01$ for example 2.

This problem is solved by different values of the step size in the x-direction h and time step size $\Delta t = 0.0001$. The computed solutions by proposed method are compared with the exact solution at the grid points and the maximum absolute errors are tabulated in Table 2. The space−time graph of the estimated solution is given in Figs. 2. The maximum absolute error of this example by method II is 1.32×10^{-8} and by method I is 3×10^{-9}.

Example3:
We consider Eq.(1) with $c = 1$, $f_1(x) = sin(x)$, $p_1(t) = 0$ and $q_1(t) = -e^{-t}sin(1)$. The exact solution for this problem is

$$u(x,t) = e^{-t}\sin(x).$$

Table 3. $Absolute\ error for Example.3$

x_i	t_j	$Method I$	$Method II$
0.1	0.0003	1×10^{-10}	1×10^{-10}
0.1	0.0005	1×10^{-10}	1×10^{-10}
0.15	0.0003	2×10^{-11}	2×10^{-11}
0.15	0.0005	4×10^{-11}	4×10^{-11}
0.2	0.0003	1×10^{-10}	1×10^{-10}
0.2	0.0005	2×10^{-10}	1×10^{-10}

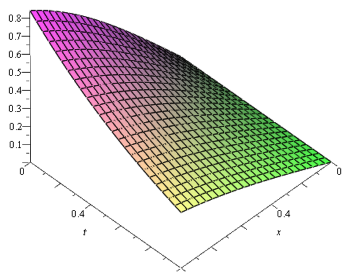

Figure 3. Space-Time graph of the solution up to t=1 s, with $c = 1$, $t = 0.0001$ and $h = 0.01$ for example 3.

This problem is solved by different values of the step size in the x-direction h and time step size $\Delta t = 0.0001$. The computed solutions by proposed method are compared with the exact solution at the grid points and the maximum absolute errors are tabulated in Table 3. The space−time graph of the estimated solution is given in Figs. 3. The maximum absolute error of this example by method II is 1×10^{-9} and by method I is 2.1×10^{-9}.

In the following examples we apply proposed method to the wave equation.

Example4:
We consider Eq.(4) with $c = 1$, $f_2(x) = 0$, $f_3(x) = \pi cos(\pi x)$, $p_2(t) = sin(\pi t)$ and

Table 4. $Absolute\ error for Example.4$

x_i	t_j	$Method I$	$Method II$
0.05	0.03	9×10^{-12}	3×10^{-12}
0.05	0.05	1.6×10^{-11}	7×10^{-12}
0.1	0.03	1.2×10^{-11}	1.7×10^{-11}
0.1	0.05	1.7×10^{-11}	1×10^{-12}
0.2	0.03	7×10^{-12}	1×10^{-12}
0.2	0.05	1×10^{-11}	1.1×10^{-11}

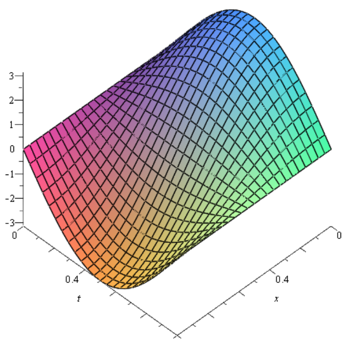

Figure 4. Space-Time graph of the solution up to t=1 s, with $c = 1$, $t = 0.01$ and $h = 0.01$ for example 4.

$q_2(t) = -sin(\pi t)$. The exact solution for this problem is

$$u\,(x, t) = \pi\ \cos\,(\pi\, x)\sin\,(\pi\, t)\,.$$

This problem is solved by different values of the step size in the x-direction h and time step size $\Delta t = 0.01$. The computed solutions by proposed method are compared with the exact solution at the grid points and the maximum absolute errors are tabulated in Table 4. The space−time graph of the estimated solution is given in Figs. 4. The maximum absolute error of this example by method II is 8.7×10^{-10} and by method I is 1.2×10^{-9}.

Example5:
We consider Eq.(4) with $c = 1$, $f_2(x) = cos(\pi x)$, $f_3(x) = 0$, $p_2(t) = cos(\pi t)$ and

Table 5. $Absolute\ error for Example.5$

x_i	t_j	$Method I$	$Method II$
0.05	0.0003	5×10^{-10}	1×10^{-9}
0.05	0.0005	1×10^{-10}	1×10^{-9}
0.1	0.0003	2.1×10^{-9}	7×10^{-10}
0.1	0.0005	9.7×10^{-9}	4×10^{-10}
0.2	0.0003	6.2×10^{-9}	1×10^{-10}
0.2	0.0005	1.3×10^{-8}	1×10^{-10}

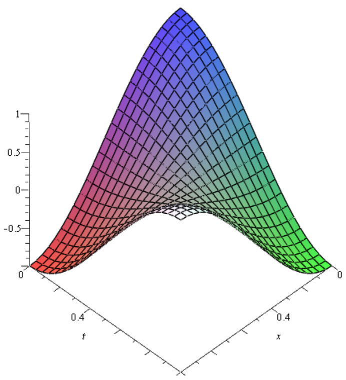

Figure 5. Space-Time graph of the solution up to t=1 s, with $c = 1$, $t = 0.0001$ and $h = 0.01$ for example 5.

$q_2(t) = -cos(\pi t)$. The exact solution for this problem is

$$u(x,t) = \frac{1}{2} \cos(\pi\ (x+t)) + \frac{1}{2} \cos(\pi\ (x-t)).$$

This problem is solved by different values of the step size in the x-direction h and time step size $\Delta t = 0.0001$. The computed solutions by proposed method are compared with the exact solution at the grid points and the maximum absolute errors are tabulated in Table 5. The space−time graph of the estimated solution is given in Figs. 5. The maximum absolute error of this example by method II is 6.5×10^{-9} and by method I is 3.84×10^{-8}.

7. Conclusion

In this paper, we constructed a three time-level spline-difference scheme for the one-dimensional heat and wave equation. Finite difference approximations for the time direction and quintic spline in space direction are used, our presented scheme is of order $O(h^8+h^4k^4)$. We solved five examples to discuss the accuracy of the method. $L\infty$ have been used for errors which are tabulated in tables. These computational results show that our proposed algorithm is effective and accurate.

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