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# Approximation Solution of Two-Dimensional Linear Stochastic Fredholm Integral Equation by Applying the Haar Wavelet

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**Abstract.**In this paper, we introduce an efficient method based on Haar wavelet to approximate a solution for the two-dimensional linear stochastic Fredholm integral equation. We also give an example to demonstrate the accuracy of the method.

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**Index to information contained in this paper**



## 1. Introduction

As we know, two dimensional ordinary integral equations provide an important tool for modeling a numerous problems in engineering and science [6, 16]. The second kind of two-dimensional integral equations appear in nonhomogeneous elasticity and electrostatics, the Dorboux problem, contact problems for bodies with complex properties, radio wave propagation, the elastic problem of axial translation of a rigid elliptical disc-inclusion, various physical and mechanical and biological problems [1, 11, 4, 10, 20, 23, 25, 28]. Some numerical schemes have been inspected for resolvent of two-dimensional ordinary integral equations by several probers. Computational complexity of mathematical operations is the most important obstacle for solving ordinary integral equations in higher dimensionas.

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These include the Nystrom method, collocation method, Gauss product quadrature rule method, Galerkin method, using triangular fuctions, Legender polynomial method, differential transform method, meshless method, Bernstein polynomials method and Haar wavelet method [5, 7, 8, 9, 12, 13, 14, 15, 19, 21, 22, 24, 29, 30]. This paper is first focused on proposing a generic framework for numerical solution of two-dimensional ordinary linear Fredholm integral equations of second kind. The use of the Haar wavelet for the numerical solution of linear integral equations has previously been discussed in [3] and references therein. In [2] a new numerical method based on Haar wavelet is introduced for solution of nonlinear one-dimensional Fredholm and Volterra integral equations. In [27] the Haar wavelet method [2] is extended to numerical solution of integro-differential equation. In [26] the Haar wavelet method [2, 27] is improved in terms of efficiency by introducing one-dimentional Haar wavelet approximation of the kernel function. The method [3] is fundamentally different from the other numerical methods based on Haar wavelet for the numerical solution of integral equations as it approximates kernel function using Haar wavelet. It's easy to show that the Fredholm integral form of the general hyperbolic differential equation [10] is given by two-dimentional integral equation

$$g(x,y) = f(x,y) + \int_0^1 \int_0^1 K_1(x,y,s,t)g(s,t)dsdt.$$

If we import statistical noise in the general hyperbolic differential equation [10], we can obtain two-dimensional linear stochastic Fredholm integral equation of the second kind, i.e.

$$g(x,y) = f(x,y) + \int_0^1 \int_0^1 K_1(x,y,s,t)g(s,t)dsdt \tag{1}$$

$$+ \int_0^1 \int_0^1 K_2(x,y,s,t)g(s,t)dB(s)dB(t)$$

$$(x,y) \in [0,1] \times [0,1]$$

where the kernels $K_1(x,y,s,t)$ and $K_2(x,y,s,t)$ in (1.1) are known functions and $f(x,y)$ is also a known function whereas $g(x,y)$ is unknown function and is called the solution of two-dimensional stochastic integral equation.

**Lemma 1.**Put $\phi(t,s) = K(x,y,s,t)g(s,t)$. Let $\phi$ be a function in $L^2([0,1]^2)$. Then there exists a sequence $\phi_n$ of off-diagonal step functions such that [18]

$$\lim_{n\to\infty} \int_a^b \int_a^b \mid \phi(t,s) - \phi_n(t,s) \mid^2 dtds = 0.$$

**Definition 1.** Let $\phi \in L^2([0,1]^2)$. Then the double Wiener-Itô integral of $\phi$ is defined as [18]

$$\int_a^b \int_a^b \phi(t,s)dB(t)dB(s) = \lim_{n\to\infty} \int_a^b \int_a^b \phi_n(t,s)dB(t)dB(s) \qquad in \quad L^2(\Omega).$$

**Theorem 1.** Let $\phi(t,s) \in L^2([a,b]^2)$. Then [18]

$$\int_a^b \int_a^b \phi(t,s)dB(t)dB(s) = 2\int_a^b \left[\int_a^t \hat{\phi}(t,s)dB(s)\right] dB(t), \tag{2}$$

where $\hat{\phi}$ is the symmetrization of $\phi$ that is defined by

$$\hat{\phi}(t,s) = \frac{1}{2}\left(\phi(t,s) + \phi(s,t)\right). \tag{3}$$

Also for the integral $\int_a^b B(t)dB(t)$ we have [18]

$$\int_a^b B(t)dB(t) = \frac{1}{2}\left\{B(b)^2 - B(a)^2 - (b-a)\right\}. \tag{4}$$

## 2. Haar Wavelets

A wavelet family $(\psi_{j,i}(y))_{j\in N, i\in Z}$ is an orthonormal subfamily of the Hilbert space $L^2(R)$ with the property that all function in the wavelet family are generated from a fixed function $\psi$ called mother wavelet through dilations and translations.The wavelet family satisfies the following relation

$$\psi_{j,i}(y) = 2^{j/2}\psi\left(2^j y - i\right).$$

For Haar wavelet family on the interval $[0,1)$ we have:

$$h_1(y) = \begin{cases} 1, & for \ \ \mathrm{y} \in [0,1) \\ 0, & otherwise, \end{cases} \tag{5}$$

and

$$h_i(y) = \begin{cases} 1, & for \mathrm{y} \in [\alpha,\beta) \\ -1, & for \mathrm{y} \in [\beta,\gamma) \\ 0, & otherwise \qquad i = 2,3,..., \end{cases} \tag{6}$$

where

$$\alpha_n = \frac{n}{m}, \qquad \beta_n = \frac{(n+0.5)}{m}, \qquad \gamma_n = \frac{(n+1)}{m};$$

$$m = 2^{\ell}, \qquad \ell = 0,1,..., \qquad n = 0,1,...,m-1.$$

The integer $\ell$ indicats the level of the wavelet and $n$ is the translation parameter. Any square integrable function $f(y)$ defined on $[0, 1)$ can be expressed as follows:

$$f(y) = \sum_{i=1}^{\infty} a_i h_i(y),$$

where $a_i$ are real constants.
For approximation aim we consider a maximum value $L$ of the integer $\ell$, level of the Haar wavelet in the above definition. The integer $L$ is then called maximum level of resolution. We also define integer $M = 2^L$. Hence for any square integrable function $f(y)$ we have a finite sum of Haar wavelets as follows:

$$f(y) \approx \sum_{i=1}^{2M} a_i h_i(y).$$

## 3. Numerical Method

In this section proposed numerical method [3] will be discussed for two-dimensional linear stochastic Fredholm integral equation of the second kind. In the first subsection, we state some results for efficient evalution of two-dimensional Haar wavelet approximations. In the second subsection, we apply these results for finding numerical solutions equation (1.1).
For Haar wavelet approximation of a function $f(x, y)$ of two real variables $x$ and $y$, we assume that the domain $0 \leqslant x, y \leqslant 1$ is divided into a grid of size $2M \times 2N$ using the following collocation points

$$x_m = \frac{m - 0.5}{2M}, m = 1, 2, ..., 2M, \tag{7}$$

$$y_n = \frac{n - 0.5}{2N}, n = 1, 2, ..., 2N. \tag{8}$$

### 3.1 Two-dimensional Haar wavelet system

A real-valued function $G(x, y)$ of two real variables $x$ and $y$ can be approximated using two-dimensional Haar wavelets basis as [3, 17]:

$$G(x, y) \approx \sum_{p=1}^{2M} \sum_{q=1}^{2N} b_{p,q} h_p(x) h_q(y). \tag{9}$$

In order to calculate the unknown coefficients $b_{i,j}$'s, the collocation points defined in Eqs. (3.1) and (3.2) are substituted in Eq. (3.3).Hence, we obtain the following $2M \times 2N$ linear system with unknowns $b_{i,j}$'s:

$$G(x_m, y_n) = \sum_{p=1}^{2M} \sum_{q=1}^{2N} b_{p,q} h_p(x_m) h_q(y_n), m = 1, 2, ..., 2M, \quad n = 1, 2, ..., 2N. \tag{10}$$

The solution of system (3.4) can be calculated from the following theorem.
**Theorem 2.** The solution of the system (3.4) is given below:

$$b_{1,1} = \frac{1}{2M \times 2N} \sum_{p=1}^{2M} \sum_{q=1}^{2N} G(x_m, y_n),$$

$$b_{i,1} = \frac{1}{\rho_1 \times 2N} \left( \sum_{p=\alpha_1}^{\beta_1} \sum_{q=1}^{2N} G(x_m, y_n) - \sum_{p=\beta_1+1}^{\gamma_1} \sum_{q=1}^{2N} G(x_m, y_n) \right) \quad , \quad i = 2, 3, ..., 2M,$$

$$b_{1,j} = \frac{1}{2M \times \rho_2} \left( \sum_{p=1}^{2M} \sum_{q=\alpha_2}^{\beta_2} G(x_m, y_n) - \sum_{p=1}^{2M} \sum_{q=\beta_2+1}^{\gamma_2} G(x_m, y_n) \right) \quad , \quad j = 2, 3, ..., 2N,$$

$$b_{i,j} = \frac{1}{\rho_1 \times \rho_2} \left( \sum_{p=\alpha_1}^{\beta_1} \sum_{q=\alpha_2}^{\beta_2} G(x_m, y_n) - \sum_{p=\alpha_1}^{\beta_1} \sum_{q=\beta_2+1}^{\gamma_2} G(x_m, y_n) - \sum_{p=\beta_1+1}^{\gamma_1} \sum_{q=\alpha_2}^{\beta_2} G(x_m, y_n) \right.$$

$$\left. + \sum_{p=\beta_1+1}^{\gamma_1} \sum_{q=\beta_2+1}^{\gamma_2} G(x_m, y_n) \right) \quad , \quad i = 2, 3, ..., 2M \quad , \quad j = 2, 3, ..., 2N,$$

where

$$\begin{aligned} \alpha_1 &= \rho_1(\sigma_1 - 1) + 1, \\ \beta_1 &= \rho_1(\sigma_1 - 1) + \tfrac{\rho_1}{2}, \\ \gamma_1 &= \rho_1 \sigma_1, \\ \rho_1 &= \tfrac{2M}{\tau_1,} \\ \sigma_1 &= i - \tau_1, \\ \tau_1 &= 2^{\lfloor \log_2(i-1) \rfloor} \end{aligned} \quad . \tag{11}$$

and similarly,

$$\begin{aligned} \alpha_2 &= \rho_2(\sigma_2 - 1) + 1, \\ \beta_2 &= \rho_2(\sigma_2 - 1) + \tfrac{\rho_2}{2}, \\ \gamma_2 &= \rho_2 \sigma_2, \\ \rho_2 &= \tfrac{2N}{\tau_2}, \\ \sigma_2 &= j - \tau_2, \\ \tau_2 &= 2^{\lfloor \log_2(j-1) \rfloor} \end{aligned} \quad . \tag{12}$$

*Proof.* See [2].

□

Consider a function $G(x, y, s, t)$ of four variables $x, y, s$ and $t$. Suppose $G(x, y, s, t)$ is approximated using two-dimensional Haar wavelet as follows [3]:

$$G(x, y, s, t) \approx \sum_{p=1}^{2M} \sum_{q=1}^{2N} b_{p,q}(x, y) h_p(s) h_q(t). \tag{13}$$

Substituting the collocation points

$$s_i = \frac{i - 0.5}{2M} \quad , \quad i = 1, 2, ..., 2M,$$

and

$$t_j = \frac{j - 0.5}{2N} \quad , \quad j = 1, 2, ..., 2N,$$

we obtain the linear system

$$G(x, y, s_i, t_j) \approx \sum_{p=1}^{2M} \sum_{q=1}^{2N} b_{p,q}(x, y) h_p(s_i) h_q(t_j) \quad , \quad i = 1, 2, ..., 2M \quad , \quad j = 1, 2, ..., 2N. \tag{14}$$

**Corollary 1.** The solution of the system (3.8) for any value of $x.y \in [0, 1]$ is given as follows [3]:

$$b_{1,1}(x, y) = \frac{1}{2M \times 2N} \sum_{p=1}^{2M} \sum_{q=1}^{2N} G(x, y, s_p, t_q),$$

$$b_{i,1}(x, y) = \frac{1}{\rho_1 \times 2N} \left( \sum_{p=\alpha_1}^{\beta_1} \sum_{q=1}^{2N} G(x, y, s_p, t_q) - \sum_{p=\beta_1+1}^{\gamma_1} \sum_{q=1}^{2N} G(x, y, s_p, t_q) \right), i = 2, 3, ..., 2M,$$

$$b_{1,j}(x, y) = \frac{1}{2M \times \rho_2} \left( \sum_{p=1}^{2M} \sum_{q=\alpha_2}^{\beta_2} G(x, y, s_p, t_q) - \sum_{p=1}^{2M} \sum_{q=\beta_2+1}^{\gamma_2} G(x, y, s_p, t_q) \right), j = 2, 3, ..., 2N,$$

$$b_{i,j}(x, y) = \frac{1}{\rho_1 \times \rho_2} \left( \sum_{p=\alpha_1}^{\beta_1} \sum_{q=\alpha_2}^{\beta_2} G(x, y, s_p, t_q) - \sum_{p=\alpha_1}^{\beta_1} \sum_{q=\beta_2+1}^{\gamma_2} G(x, y, s_p, t_q) \right.$$

$$\left. - \sum_{p=\beta_1+1}^{\gamma_1} \sum_{q=\alpha_2}^{\beta_2} G(x, y, s_p, t_q) + \sum_{p=\beta_1+1}^{\gamma_1} \sum_{q=\beta_2+1}^{\gamma_2} G(x, y, s_p, t_q) \right),$$

$$i = 2, 3, ..., 2M \quad , \quad j = 2, 3, ..., 2N,$$

where $\alpha_1, \beta_1, \gamma_1$ and $\rho_1$ are defined as in Eq. (3.5) and $\alpha_2, \beta_2, \gamma_2$ and $\rho_2$ are defined as in Eq. (3.6).

**Corollary 2.** Suppose a function $G(x, y)$ of two variables $x$ and $y$ is approximated using Haar wavelet approximation given in Eq. (3.3). Suppose further that $G(x, y)$ is known at collocation points $(x_m, y_m)$, $m = 1, 2, ..., 2M, n = 1, 2, ..., 2N$. Then the approximate value of the function $G(x, y)$ at any other point of the domain can be calculated as follows [3]:

$$G(x,y) = \frac{1}{2M \times 2N} \sum_{p=1}^{2M} \sum_{q=1}^{2N} G(x_m, y_m) h_1(x) h_1(y)$$

$$+ \sum_{i=1}^{2M} \frac{1}{\rho_1 \times 2N} \left( \sum_{p=\alpha_1}^{\beta_1} \sum_{q=1}^{2N} G(x_m, y_m) - \sum_{p=\beta_1+1}^{\gamma_1} \sum_{q=1}^{2N} G(x_m, y_m) \right) h_i(x) h_1(y)$$

$$+ \sum_{j=1}^{2N} \frac{1}{2M \times \rho_2} \left( \sum_{p=1}^{2M} \sum_{q=\alpha_2}^{\beta_2} G(x_m, y_m) - \sum_{p=1}^{2M} \sum_{q=\beta_2+1}^{\gamma_2} G(x_m, y_m) \right) h_1(x) h_j(y)$$

$$+ \sum_{i=1}^{2M} \sum_{j=1}^{2N} \frac{1}{\rho_1 \rho_2} \left( \sum_{p=\alpha_1}^{\beta_1} \sum_{q=\alpha_2}^{\beta_2} G(x_m, y_m) - \sum_{p=\alpha_1}^{\beta_1} \sum_{q=\beta_2+1}^{\gamma_2} F(x_m, y_m) \right.$$

$$\left. - \sum_{p=\beta_1+1}^{\gamma_1} \sum_{q=\alpha_2}^{\beta_2} G(x_m, y_m) + \sum_{p=\beta_1+1}^{\gamma_1} \sum_{q=\beta_2+1}^{\gamma_2} G(x_m, y_m) \right) h_i(x) h_j(y),$$

where $\alpha_1, \beta_1, \gamma_1$ and $\rho_1$ are defined as in Eq. (3.5) and $\alpha_2, \beta_2, \gamma_2$ and $\rho_2$ are defined as in Eq. (3.6).

### 3.2 Two-dimensional linear stochastic Fredholm integral equation

Consider the two-dimensional linear stochastic Fredholm integral equation (1.1). First we define:

$$K_2(x, y, s, t) g(s, t) = \phi(t, s),$$

afterward from (1.3) we have

$$\hat{\phi}(t, s) = \frac{1}{2} \left\{ K_2(x, y, t, s) g(t, s) + K_2(x, y, s, t) g(s, t) \right\}.$$

Assume that the function $K(x, y, s, t)g(s, t)$ is approximated using two-dimensional Haar wavelet as follows:

$$K_1(x, y, s, t) g(s, t) \approx \sum_{i=1}^{2M} \sum_{j=1}^{2N} b_{i,j}(x, y) h_i(s) h_j(t), \tag{15}$$

$$\hat{\phi}(t,s) \approx \sum_{i=1}^{2M}\sum_{j=1}^{2N} c_{i,j}(x,y)h_i(s)h_j(t). \tag{16}$$

With this approximation and using Eq. (1.2) we can write Eq. (1.1) as follows:

$$g(x,y) = f(x,y) + \int_0^1\int_0^1 \sum_{i=1}^{2M}\sum_{j=1}^{2N} b_{i,j}(x,y)h_i(s)h_j(t)dsdt \tag{17}$$

$$+2\int_0^1 \left[\int_0^t \sum_{i=1}^{2M}\sum_{j=1}^{2N} c_{i,j}(x,y)h_i(s)h_j(t)dB(s)\right] dB(t).$$

Eq. (3.11) can be written in a more compact form using the notations introduced in equations (2.1) and (2.2) and is given as follows:

$$g(x,y) = f(x,y) + b_{1,1}(x,y) + 2\left(c_{1,1}(x,y)\times Ito1(1) + \sum_{j=2}^{2N} c_{1,j}(x,y)\times Ito2(j)\right.$$

$$\left. + \sum_{i=2}^{2M} c_{i,1}(x,y)\times Ito1(i) + \sum_{i=2}^{2M}\sum_{j=2}^{2N} c_{i,j}(x,y)\times Ito3(i,j)\right),$$

where in recent equation from Eq. (1.4) we have:

$$Ito1(1) = \int_0^1 h_1(t)\left[\int_0^t h_1(s)dB(s)\right]dB(t) = \int_0^1 B(t)dB(t) = \frac{B^2(1)}{2} - \frac{1}{2},$$

$$Ito1(i) = \int_0^1 h_1(t)\left[\int_0^t h_i(s)dB(s)\right]dB(t) = \int_0^1\left[\int_{\alpha_i}^{\beta_i} dB(s) - \int_{\beta_i}^{\gamma_i} dB(s)\right]dB(t)$$

$$= \left(2B(\beta_i) - B(\alpha_i) - B(\gamma_i)\right)B(1),$$

$$Ito1(j) = \int_0^1 h_j(t)\left[\int_0^t h_1(s)dB(s)\right]dB(t) = \int_0^1 h_j(t)B(t)dB(t)$$

$$= \int_{\alpha_j}^{\beta_j} B(t)dB(t) - \int_{\beta_j}^{\gamma_j} B(t)dB(t) = \frac{2mB^2(\beta_j) - B^2(\alpha_j) - B^2(\gamma_j) + 1}{2m},$$

and

$$Ito3(i,j) = \int_0^1 h_j(t)\left[\int_0^t h_i(s)dB(s)\right]dB(t)$$

$$= \int_0^1 h_j(t) \left[ \int_{\alpha_i}^{\beta_i} dB(s) - \int_{\beta_i}^{\gamma_i} dB(s) \right] dB(t)$$

$$= \left[2B(\beta_i) - B(\alpha_i) - B(\gamma_i)\right] \left[2B(\beta_j) - B(\alpha_j) - B(\gamma_j)\right].$$

Substituting the collocation points given in (3.1) and (3.2), we obtain the following system of equations:

$$g(x_m, y_n) = f(x_m, y_n) + b_{1,1}(x_m, y_n) + 2 \Bigg( c_{1,1}(x_m, y_n) \times Ito1(1) + \sum_{j=2}^{2N} c_{1,j}(x_m, y_n)$$

$$\times Ito2(j) + \sum_{i=2}^{2M} c_{i,1}(x_m, y_n) \times Ito1(i) + \sum_{i=2}^{2M} \sum_{j=2}^{2N} c_{i,j}(x_m, y_n) \times Ito3(i, j) \Bigg).$$

Now $b_{i,j}$ , $i = 1, 2, ..., 2M$ , $j = 1, 2, ..., 2N$ and similarly $c_{i,j}$ , $i = 1, 2, ..., 2M$ , $j = 1, 2, ..., 2N$ can be replaced with their expressions given in Corollary 1 and the following system of equations is obtained:

$$g(x_m, y_n) = f(x_m, y_n) + \frac{1}{2M \times 2N} \sum_{p=1}^{2M} \sum_{q=1}^{2N} K_1(x_m, y_n, s_p, t_q) g(s_p, t_q) \tag{18}$$

$$+ \left[ \frac{1}{M \times 2N} \sum_{p=1}^{2M} \sum_{q=1}^{2N} \hat{\phi}(t_q, s_p) \right] \times Ito1(1)$$

$$+ \sum_{j=2}^{2N} \frac{1}{M \times \rho_2} \left[ \sum_{p=1}^{2M} \sum_{q=\alpha_2}^{\beta_2} \hat{\phi}(t_q, s_p) - \sum_{p=1}^{2M} \sum_{q=\beta_2+1}^{\gamma_2} \hat{\phi}(t_q, s_p) \right] \times Ito2(j)$$

$$+ \sum_{i=2}^{2M} \frac{1}{\rho_1 \times N} \left[ \sum_{p=\alpha_1}^{\beta_1} \sum_{q=1}^{2N} \hat{\phi}(t_q, s_p) - \sum_{p=\beta_1+1}^{\gamma_1} \sum_{q=1}^{2N} \hat{\phi}(t_q, s_p) \right] \times Ito1(i)$$

$$+2 \sum_{i=2}^{2M} \sum_{j=2}^{2N} \frac{1}{\rho_1 \times \rho_2} \Bigg[ \sum_{p=\alpha_1}^{\beta_1} \sum_{q=\alpha_2}^{\beta_2} \hat{\phi}(t_q, s_p) - \sum_{p=\alpha_1}^{\beta_1} \sum_{q=\beta_2+1}^{\gamma_2} \hat{\phi}(t_q, s_p)$$

$$- \sum_{p=\beta_1+1}^{\gamma_1} \sum_{q=\alpha_2}^{\beta_2} \hat{\phi}(t_q, s_p) + \sum_{p=\beta_1+1}^{\gamma_1} \sum_{q=\beta_2+1}^{\gamma_2} \hat{\phi}(t_q, s_p) \Bigg] \times Ito3(i, j).$$

Table 1. The solution mean with %95 confidence interval for above example

| $L$ | $M$ | $2M$ | $(x,y)$ | $\bar{g}(x,y)$ | %95 Confidence Interval $L$ | $U$ |
|---|---|---|---|---|---|---|
| 0 | 1 | 2 | $(0.25, 0.25)$ | 0.394433 | 0.392491 | 0.396374 |
| | | | $(0.25, 0.75)$ | 0.881874 | 0.879448 | 0.884299 |
| | | | $(0.75, 0.25)$ | 0.882179 | 0.879495 | 0.884864 |
| 1 | 2 | 4 | $(0.125, 0.625)$ | 0.626416 | 0.621964 | 0.630868 |
| | | | $(0.375, 0.375)$ | 0.626752 | 0.619833 | 0.633671 |
| | | | $(0.875, 0.375)$ | 1.06476 | 1.03276 | 1.09675 |
| 2 | 4 | 8 | $(0.0625, 0.8125)$ | 0.625749 | 0.620075 | 0.631423 |
| | | | $(0.4375, 0.4375)$ | 0.616593 | 0.611904 | 0.621282 |
| | | | $(0.9375, 0.1875)$ | 0.854036 | 0.834678 | 0.873394 |
| 3 | 8 | 16 | $(0.03125, 0.84375)$ | 0.532988 | 0.522597 | 0.543379 |
| | | | $(0.40625, 0.40625)$ | 0.459968 | 0.449544 | 0.470391 |
| | | | $(0.9375, 0.1875)$ | 0.695789 | 0.686792 | 0.704786 |
| 4 | 16 | 32 | $(0.015625, 0.515625)$ | 0.0968562 | 0.0897225 | 0.10399 |
| | | | $(0.765625, 0.765625)$ | 0.87951 | 0.868732 | 0.890288 |
| | | | $(0.984375, 0.234375)$ | 0.746029 | 0.735251 | 0.756807 |

Eq. (3.12) represents $2M \times 2N$ system which can be solved using either prevalent methods for solving linear systems. The solution of this system gives values of $g(x,y)$ at the collocation points. The values of $g(x,y)$ at points other than collocation points can be calculated using Corollary 2.

## 4. Numerical Example

In this section, the numerical example is given to demonstrate the applicability and accuracy of our method. Consider the following linear two dimensional stochastic Fredholm integral equation of second kind:

$$u(x,y) = f(x,y) + \int_0^1 \int_0^1 (x+y+t-s)g(s,t)dsdt + \int_0^1 \int_0^1 (x+y+t+s)g(s,t)dB(s)dB(t)$$

where

$$f(x,y) = x + y - \frac{1}{12}xy(x^3 + 4x^2y + 4xy^2 + y^3).$$

The solution mean with confidence interval at the collocation points 1000 iterative of system (3.12) is shown in Table 1. In Figs. 1, three-dimensional graph of the approximate solution for level $L = 4$ is shown.

Figure 1. Approximate solution plot ($L = 4$).

## 5. Conclusion

The numerical solution of two-dimensional stochastic integral equations because of the randomness is very difficult or sometimes impossible. In this paper, we have successfully developed Haar wavelets numerical method for approximate a solution of two-dimensional linear stochastic Fredholm integral equations. The example confirm that the method is considerably fast and highly accurate as sometimes lead to exact solution. Although, theoretically for getting higher accuracy we can set the method with larger values of M and N and also larger of the degree of approximation, p and q, but it leads to solving MN linear systems of size $pq \times pq$, that have its difficulties. The method can be improved to be more accurate by using other numerical methods. Mathematica has been used for computations.

## References


[1] Ahmadi. S, Application of partial differential equations in snow mechanics, Mathematical Modelling and Computations, (2011) 189-194.

[2] Aziz. I and Siraj-ul-Islam, New algorithms for numerical solution of nonlinear Fredholm and Volterra integral equations using Haar wavelets, J. Comp. Appl. Math, **239** (2013) 333-345.

[3] Aziz. I, Siraj-ul-Islam and F. Khan, A new method based on Haar wavelet for numerical solution of two-dimensional nonlinear integral equations, J. Comp. Appl. Math, **272** (2014) 70-80.

[4] Aleksandrov. V. M and Manzhirov. A. V, Two-dimentional integral equations in applied mechanics of deformable solids, J. Appl. Mech. Tech. Phys, **5** (1987) 146-152.

[5] Assari. P, Adibi. H and Dehghal. M, A meshless method for solving nonlinear two-dimensional integral equations of the second kind on non-rectangular domains using radial basis functions with error analysis, J. Comp. Appl. Math, **239** (2013) 72-92.

[6] Atkinson. K. E, The numerical solution of integral equations of the second kind, Cambridge University Press, (1997).

[7] Babolian. E, Maleknejad. K, Roodaki. M and Almasieh. H, Two dimensional triangular functions and their applications to nonlinear 2d Volterra-Fredholm equations, Comp. Math. App, **60** (2010) 1711-1722.

[8] Bazm. S and Babolian. E, Numerical solution of nonlinear two-dimensional Fredholm integral equations of the second kind using Gauss product quadrature rules, Commun. Nonlinear Sci. Numer. Simult, **17** (2012) 1215-1223.

[9] Brunner. H, Collocation methods for Volterra integral and related functional equations, Cambridge University Press, (2004).

[10] Dobner. H. J, Bounds for the solution of hyperbolic problems, Computing, **38** (1987) 209-218.

[11] Elkhayyari. O and Lamnii. A, Numerical solutions of second order boundary value problem by using hyperbolic uniform b-splines of order 4, Mathematical Modelling and Computations, (2014) 25-36.

[12] Guoqiang. H, Itayami. H, Sugihara. K and Jiong. W, Extrapolation method of iterated collocation solution for two-dimentional nonlinear Volterra integral equations, Appl. Math. Comput, **112** (2000) 49-61.

[13] Guoqiang. H and Jiong. W, Extrapolation of nystrom solution for two dimentional nonlinear Fredholm integral equations, J. Comp. App. Math, **134** (2001) 259-268.

[14] Han. G and Wang. R, Richardson extrapolation of iterated discrete Galerkin solution for two-dimensional Fredholm integral equations, J. Comp. App. Math, **139** (2002) 49-63.

[15] Hosseini Shekarabi. F, Maleknejad. K and Ezzati. R, Application of two-dimensional Bernstein polynomials for solving mixed Volterra-Fredholm integral equations, African Mathematical Union and Springer-Verlag Berlin Heidelberg, DOI 10. 1007/s 13370-014-0283-6, (2014).

[16] Jerri. A. J, Introduction to integral equations with applications, John Wiley and Sons, INC, (1999).

[17] Keinert. F, Wavelets and Multiwavelets, A Crc Press Company Boca Raton London New York Washington, D. C, (2004).

[18] Kuo, Hui-Hsiung, Introduction to stochastic integration, Springer Science+Business Media, Inc, (2006).

[19] Maleknejad. K and JafariBehbahani. Z, Application of two-dimensional triangular functions for solving nonlinear class of mixed Volterra-Fredholm integral equations, Math. Comp. Mode, **55** (2012) 1833-1844.

[20] Manzhirov. A. V, Contact problems of the interaction between viscoelastic foundations subject to ageing and systems of stamps not applied simultaneously, Prikl. Matem. Mekhan, **4** (1987) 523-535.

[21] Nemati. S and Ordokhani. Y, Solving nonlinear two-dimensional Volterra integral equations of the first-kind using the bivariate shifted legendre functions, Mathematical Modelling and Computations, (2015) 219- 230.

[22] Nemati. S, Lima. P and Ordokhani. Y, Numerical solution of a class of two-dimensional nonlinear Volterra integral equations using legender polynomials, J. Comp. Appl. Math, **242** (2013) 53-69.

[23] Rahman. M, "A rigid elliptical disc-inclusion in an elastic solid", subject to a polynomial normal shift, J. Elasticity, **66** (2002) 207-235.

[24] Reihani. M. H and Abadi. Z, Rationalized Haar functions method for solving Fredholm and Volterra integral equations, J. Comp. Appl. Math, **200** (2007) 12-20.

[25] Sankar. T. S and Fabrikant. V. I, Investigations of a two-dimentional integral equation in the theory of elasticity and electrostatics, J. Mec. Theor. Appl, **2** (1983) 285-299.

[26] Siraj-ul-Islam, Aziz. I and Al-Fhaid. A, An improved method based on Haar wavelets for numerical solution of nonlinear and integro-differential equations of first and higher orders, J. Comp. Appl. Math, **260** (2014) 449-469.

[27] Siraj-ul-Islam, Aziz. I and Fayyaz. M, A new approach for numerical solution of integro-differential equations via Haar wavelets, Int, J. Comp. Math, **90** (2013) 1971-1989.

[28] Soloviev. O. V, Low-frequency radio wave propagation in the earth-ionosphere waveguide disturbed by a large-scale three-dimensional irregularity, Radiophysics and Quantum Electronics, **41** (1998) 392-402.

[29] Tari. A, Rahimi. M, Shahmorad. S and Talati. F, Solving a class of two-dimensional linear and nonlinear Volterra integral equations by the differential transform method, J. Comp. Appl. Math, **228** (2009) 70-76.

[30] Xie. W and Lin. F. R, A fast numerical solution method for two dimensional Fredholm integral equations of the second kind, App. Num. Math, **59** (2009) 1709-1719.