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The Entropies of the Sequences of Fuzzy Sets and Applications of Entropy to Cardiography

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Abstract. In this paper, firstly we have introduced entropy of sequences of fuzzy sets and given some theorems about it. Secondly, the waves P and T which appear in electrocardiograms were transferred to fuzzy sets, by using definition of entropy for sequences of fuzzy sets, and some numerical values were obtained for sequences of waves P and T. Thus any person can make a medical predictions for some cardiac problems using numerical values.

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1. Background, Notations and New Definitions

It is known that the theoretical and practical applications of fuzzy set have increased daily after Zadeh's paper, [23]. There are many theoretical applications of fuzzy sets, for example, you can see [3], [7, 8], [14] and [24]. Not just theoretical, the practical applications have been studied in many fields [1], [19], [12] and this list is too much than given here.

Now, together of our new definitions, we will give some background on fuzzy sets and entropy of the fuzzy sets in this section.

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Let $\mathscr{X}$ be nonempty set. According to Zadeh, a fuzzy subset of $\mathscr{X}$ is a nonempty subset $\{(x, u(x)) : x \in \mathscr{X}\}$ of $\mathscr{X} \times [0,1]$ for some function $u : \mathscr{X} \to [0,1]$, [7]. The function u is called membership function of the fuzzy set u.

Furthermore, we know that shape similarity of the membership functions does not reflect the conception of itself, but it will be used for examining the context of the membership functions. Whether a particular shape is suitable or not can be determined only in the context of a particular application. However, that many applications are not overly sensitive to variations in the shape. In such cases, it is convenient to use a simple shape, such as the triangular shape of membership function. Let us define fuzzy set u on the set $\mathbb{R}$ with membership function as follows:

$$u(x) = \begin{cases} \frac{h_u}{u_1-u_0}(x-u_0),\ x \in [u_0, u_1) \\ \frac{-h_u}{u_2-u_1}(x-u_1)+h_u,\ x \in (u_1, u_2] \\ 0,\ others \end{cases}, \tag{1}$$

where the notations h_u denotes height of the fuzzy sets u. For brief, we write triple $(u_0, u_1 : h_u, u_2)$ for fuzzy set u. Notation $\mathscr{F}$ be the set of the all fuzzy sets in the form $u = (u_0, u_1 : h_u, u_2)$ on the $\mathbb{R}$. Using to the representation $(u_0, u_1 : h_u, u_2)$, we will construct new algebraic structure for the set $\mathscr{F}$, (see Definition 2.6).

Define the function S as follows,

$$S : \mathscr{F} \times \mathscr{F} \to \mathbb{R},\quad S(u,v) = \frac{\min\{h_u, h_v\}}{\max\{h_u, h_v\}}[1 - \frac{1}{3}\sum_{k=0}^{2}|u_k - v_k|], \tag{2}$$

where the notations h_u and h_v denote maximum height of the fuzzy sets u and v.

Definition 1.1 *The function S is called similarity degree between the fuzzy sets u and v. If $S(u,v) = 1$ then we say that u is full similar to v or vice versa, we say that v is completely similar to u. If $0 < S(u,v) < 1$ then we say that the fuzzy number u is S- similar to the fuzzy number v (or the fuzzy number v is S- similar to the fuzzy number u), if $S(u,v) \leqslant 0$ we say that, u is not similar to v.*

Similar definitions can be found in [18] and [22].

Let us define the following set.

$$w(\mathscr{F}) = \{(u_k) \mid u : \mathbb{N} \to \mathscr{F}, u(k) = (u^k) = ((u_0^k, u_1^k : h_{u^k}, u_2^k))\}. \tag{3}$$

Any element of the set $w(\mathscr{F})$ is called sequences of fuzzy sets, where $u_0^k, u_1^k, u_2^k \in \mathbb{R}$ and $u_0^k \leqslant u_1^k \leqslant u_2^k$ and the mean of notation $u_1^k : h_{u^k}$ is the k^{th} term of the sequence (u^k) takes highest membership degree at u_1^k and this membership degree is equal to h_{u^k} . If for all $k \in \mathbb{N}$, $h_{u^k} = 1$ then the set $w(\mathscr{F})$ turns into sequence set of fuzzy numbers and if $u_0^k = u_1^k = u_2^k$ and $h_{u_1^k} = 1$ the set $w(\mathscr{F})$ turns into ordinary sequence space of the real numbers, respectively.

An another important class of the sequence set of the fuzzy sets is defined by

$$\varphi(\mathscr{F}) = \{(u_k) \in w(\mathscr{F}) \mid\ \exists k_0 \in \mathbb{N}, \forall k \geqslant k_0 : u^k = 0\}. \tag{4}$$

Clearly, the sequences of fuzzy sets can be obtained by fuzzification of the term by term of sequence of real numbers with a suitable method.

Let us define the function $\mathscr{S}$ as follows:

$$\mathscr{S} : w(\mathscr{F}) \times w(\mathscr{F}) \to \mathbb{R}, \mathscr{S}(u_n, v_n) = \frac{\inf\{h_u, h_v\}}{\sup\{h_u, h_v\}}[1 - \frac{1}{3}\lim_n \sum_{k=0}^{2}|u_k^n - v_k^n|] = \lambda \tag{5}$$

DEFINITION 1.2 *The function $\mathscr{S}$ is called similarity degree between sequences of fuzzy sets (u_n) and (v_n). If $\mathscr{S}(u_n, v_n) = 1$ then we say that (u_n) is completely similar to the sequence (v_n), if $0 < \mathscr{S}(u_n, v_n) = \lambda < 1$ then we say that the sequence (u_n) is λ- similar to the sequence (v_n), if $\lambda \leqslant 0$ we say that, (u_n) is not similar to (v_n).*

In the fuzzy set theory, the fuzziness of a fuzzy set is an important matter and there are many method to measure the fuzziness of a fuzzy set. At first, the fuzziness was thought to be the distance between fuzzy set and its nearest nonfuzzy set. Later, the entropy was used instead of of fuzziness [13] and has received attention, recently [21]. Well, then what is the entropy?

DEFINITION 1.3 *[25] Let $u \in \mathscr{F}$ and $u(x)$ be the membership function of the fuzzy set u and consider the function $H : \mathscr{F} \to \mathbb{R}^+$. If the function H satisfies conditions below,*

(1) $H(u) = 0$ iff u is crisp set,
(2) $H(u)$ has a unique maximum, if $u(x) = \frac{1}{2}$, for all $x \in \mathbb{R}$
(3) For $u, v \in \mathscr{F}$, if $v(x) \leqslant u(x)$ for $u(x) \leqslant \frac{1}{2}$ and $u(x) \leqslant v(x)$ for $u(x) \geqslant \frac{1}{2}$ then $H(u) \geqslant H(v)$,
(4) $H(u^c) = H(u)$, where u^c is the complement of the fuzzy set u

then the $H(u)$ is called entropy of the fuzzy set u.

Let suppose that $u = u(x)$ be membership function of the fuzzy set u and the function $h : [0, 1] \to [0, 1]$ satisfies the following properties:

(1) Monotonically increasing at $[0, \frac{1}{2}]$ and decreasing $[\frac{1}{2}, 1]$,
(2) $h(x) = 0$ if $x = 0$ and $h(x) = 1$ if $x = \frac{1}{2}$.

The function h is called entropy function and the equality $H(u(x)) = h(u(x))$ holds for $x \in \mathbb{R}$. Some well known entropy functions are given as follows:
$h_1(x) = 4x(1-x)$, $h_2(x) = -x \ln x - (1-x)\ln(1-x)$, $h_3(x) = \min\{2x, 2-2x\}$ and

$$h_4(x) = \begin{cases} 2x, \, x \in [0, \frac{1}{2}] \\ 2(1-x), \, x \in [\frac{1}{2}, 1] \end{cases}.$$

Note that the function h_1 is the logistic function, h_2 is called Shannon function and h_3 is the tent function.

2. Some Basic Theorems and Applications About Entropies of the Sequence of Fuzzy Sets

We will give some basic theorems about the entropy of the sequence of fuzzy sets, in this section and suppose that the sequences of fuzzy sets are in the form
$u = ((u_0^k, u_1^k : h_{u_1^k}, u_2^k))$ and $u_0^k \leqslant u_1^k \leqslant u_2^k$ for all $k \in \mathbb{N}$.

Let $\mathscr{X}$ be a continuous universal set. The total entropy of the fuzzy set u on the $\mathscr{X}$ is defined

$$e(u) = \int_{x \in \mathscr{X}} h(u(x))p(x)dx \tag{6}$$

where $p(x)$ is the probability density function of the available data in $\mathscr{X}$ [15], [16]. If we take $p(x) = 1$ in the (6) then the $e(u)$ is called entropy of the fuzzy set u. It is known that the value of $e(u)$ is depend on support of the fuzzy set u. Let u be fuzzy set on the set $\mathbb{R}$ with membership function (1), then we see that the total entropy of fuzzy set u is equal to

$$e(u) = c(2h_u - \frac{4}{3}h_u^2)\ell(u) \tag{7}$$

for $p(x) = c$ and $h = h_1$, where $\ell(u) = \max\{x - y : x, y \in \overline{\{x \in \mathbb{R} : u(x) > 0\}}\}$. We know that each fuzzy set or a fuzzy number correspond to the fuzzy thoughts in the idea of user. So, any sequence of the fuzzy sets can be seen as sequence of thoughts or sequence fuzzy information. This sequence of fuzzy information may contain an useful information or not contain an useful information. But we can use terms of this sequence to obtain meaningful information from this sequence.

Now, let us give some new definitions as follows:

Definition 2.1 *Let h be an entropy function, (u^k) be a sequence of fuzzy sets (or fuzzy thought) and $p_k(x)$ be probability density function of the available data in $\mathbb{R}$ for every $k \in \mathbb{N}$. Then sequence*

$$e(u^k) = \int_{x\in\mathbb{R}} h(u^k(x))p_k(x)dx \tag{8}$$

is called total entropy sequence of the fuzzy sets (u^k). If the probability density function $p_k(x) = 1$ is fix, for all $k \in \mathbb{N}$, then the (8) *is called entropy sequence of the fuzzy sets $u = (u^k)$.*

Let us suppose that $u = (u^k) \in w(\mathscr{F})$, $p_k(x) = c_k \in (0, 1]$. If we take $h(u) = h_1(u)$ then the (8) turns to

$$e(u^k) = (c_k(2h_{u^k} - \frac{4}{3}h_{u^k}^2)\ell(u^k)), \tag{9}$$

here and other places in the text, the notation $2h_{u^k}^2$ denotes second power of the h_{u^k}. If we choose the probability density functions $p_k(x) = c \in (0, 1]$ for all $k \in \mathbb{N}$ and $h_{u^k} = 1$ for all $k \in \mathbb{N}$ in the (9) then we see that $e(u^k) = \frac{2}{3}c\ell(u^k)$.

Let us suppose that $u = (u^k)$ be sequences of the fuzzy numbers (that is $h_{u^k} = 1$), $h(u) = h_1(u)$ and $p_k(x) = c_k = 1 \in (0, 1]$ for all $k \in \mathbb{N}$. Then the entropy $e(u^k)$ of the sequence of fuzzy numbers (u^k) is equal to

$$e(u^k) = \frac{2}{3}\ell(u^k). \tag{10}$$

Clearly, if $\ell(u^k) = 0$ for every $k \in \mathbb{N}$ then the sequence (u^k) returns to sequence of real numbers. In this case the entropy of the total entropy sequence is zero for sequences of real numbers. For example, let $u = (u^k)$ be $((1, 1 : 1, 1))$, then from (10) we obtain zeros sequence. Furthermore, the entropy sequence (e_k) can not be convergent but be bounded.

Definition 2.2 *Let $\mathscr{A} = (a_{nk})$ be an lower triangular infinite matrix of real or complex numbers and*

$$\sum_k a_{nk} \int_{x\in\mathbb{R}} h(u^k(x))p_k(x)dx \to E, \quad n \to \infty. \tag{11}$$

The real number E is called total $\mathscr{A}$-entropy of the sequence (u^k) of fuzzy sets, if it exists.

DEFINITION 2.3 *Let suppose that the $u = (u^k)$ be a sequence of fuzzy sets, $p_k(x) = c_k$, $(c_k \in (0,1])$ for all $k \in \mathbb{N}$ and*

$$\lim_n \sum_k a_{nk} \int_{x\in\mathbb{R}} h(u^k(x))p_k(x)dx = \lim_n \sum_k a_{nk}c_k(2h_{u^k} - \frac{4}{3}h^2_{u^k})\ell(u^k) = E_1. \quad (12)$$

The real number E_1 is called total $\mathscr{A}$- entropy according to entropy function h and $p_k(x) = c_k$ is probability density functions of the sequence $u = (u^k)$ of fuzzy sets, and it is shown by $T_e^{\mathscr{A}}(u^k)$.

Let $n, k \in \mathbb{N}$, $\alpha > -1$, $p_k(x) = c_k$ and $\binom{n-k+\alpha-1}{n-k}$, $\binom{n+\alpha}{n}$ are binomial confidence. Let us define infinite matrices $A = (a_{nk})$ and $C^\alpha = (c^\alpha_{nk})$ as follows:

$$a_{nk} = \begin{cases} 1, \, 0 \leqslant k \leqslant n \\ 0, \, otherwise \end{cases} \quad \text{and} \quad c^\alpha_{nk} = \begin{cases} \frac{\binom{n-k+\alpha-1}{n-k}}{\binom{n+\alpha}{n}}, \, 0 \leqslant k \leqslant n \\ 0, \qquad otherwise \end{cases}.$$

If we write the matrices A and C^α instead of $\mathscr{A}$ in the expression (12) then we have

$$\lim_n \sum_{k=0}^{n} \int_{x\in\mathbb{R}} h(u^k(x))p_k(x)dx = T_e^A(u^k) \quad (13)$$

and

$$\lim_n \frac{1}{\binom{n+\alpha}{n}} \sum_{k=0}^{n} \binom{n-k+\alpha-1}{n-k} \int_{x\in\mathbb{R}} h(u^i(x))p_k(x)dx = T_e^{C^\alpha}(u^k), \quad (14)$$

respectively.

The expressions (13) and (14) are called A- total entropy and total Cesàro entropy of order α of the sequence $u = (u^k)$ of fuzzy sets, according to probability density functions $p_k(x)$, respectively. In the special case for $\alpha = 1$ and $p_k(x) = c_k$ then (14) reduces to

$$T_e^{C^1}(u^k) = \lim_n \frac{1}{n+1} \sum_{k=0}^{n} c_k(2h_{u_1^k} - \frac{4}{3}h^2_{u_1^k})\ell(u^k) \quad (15)$$

which is called Cesàro normalized entropy of order 1 (shortly, Cesàro entropy) of the sequence $u = (u^k)$ of fuzzy sets.

It is easily prove that, if

$$T_e^{C^1}(u^k) = \lim_n \frac{1}{n+1} \sum_{k=0}^{n} c_k(2h_{u_1^k} - \frac{4}{3}h^2_{u_1^k})\ell(u^k) = a$$

then

$$T_e^{C^1}(u^k) = \lim_n \frac{s}{n+r} \sum_{k=0}^{n} c_k(2h_{u_1^k} - \frac{4}{3}h^2_{u_1^k})\ell(u^k) = a$$

where $r, s \in \mathbb{R}$. For example, the Cesàro entropy of sequence $(u^k) = ((\frac{k}{k+1} - t_1, \frac{k}{k+1} : 1, \frac{k}{k+1} + t_2))$ is

$$T_e^C(u^k) = \lim_n \frac{2(t_2 + t_1)}{3(n+1)} \sum_{k=0}^{n} c_k, \tag{16}$$

where we assume that $t_1 < t_2$ and $t_1, t_2 \in \mathbb{R}$ and $h_{u^k} = 1$ for all $k \in \mathbb{N}$. If the series $\sum_k c_k$ is convergent then the value $T_e^C(u^k)$ exists every time. As a comment of the (13) and (16), we point out that we can obtain an useful information from infinite fuzzy information by a suitable method. But, the total entropy and Cesàro entropy of the sequence v defined by $v = ((v_0^k, v_1^k, v_2^k)) = ((-k, 1 : 1, k+2))$ is infinite. This means that, the sequence v does not contain any useful information for us.

Since, every real number is also a fuzzy number then we can give following corollary:

Corollary 2.4 *Let the sequence $r = (r^k)$ be a convergent or divergent sequence of real numbers. Then the all entropies of the $r = (r^k)$ are zero.*

Corollary 2.4 can be interpreted as, in the any information sequence, if the elements of information sequence are crisp information then we obtain a crisp information from this sequence.

Proposition 2.5 *If the fuzziness of the any sequence of fuzzy set is constantly increasing then the entropy is constantly grow and maybe is infinite. On the contrary if the fuzziness of the any sequence of fuzzy set is constantly decreasing then the entropy decreases and becomes 0.*

It is calculated in [20] that the entropy of any fuzzy number is $\frac{2c(u_2 - u_0)}{3}$. Therefore, in generally, if we take $h = h_1$ and $p_i(x) = c$, for every $i \in \mathbb{N}$, then entropy of the sequence of fuzzy numbers is given with (10).

Definition 2.6 *Let $u = (u_0, u_1 : h_{u_1}, u_2)$ and $v = (v_0, v_1 : h_{v_1}, v_2)$ be two fuzzy sets and define the addition and scalar multiplication as follows: $(u+v) = (u_0, u_1 : h_{u_1}, u_2) + (v_0, v_1 : h_{v_1}, v_2) = (u_0 + v_0, u_1 + v_1 : \max\{h_{u_1}, h_{v_1}\}, u_2 + v_2)$ and $\alpha u = \alpha(u_0, u_1 : h_{u_1}, u_2) = (\alpha u_0, \alpha u_1 : h_{u_1}, \alpha u_2)$ if $\alpha \geqslant 0$ and $\alpha u = \alpha(u_0, u_1 : h_{u_1}, u_2) = (\alpha u_2, \alpha u_1 : h_{u_1}, \alpha u_0)$ if $\alpha < 0$ for $\alpha \in \mathbb{R}$.*

Let us suppose that $(u_k) = ((u_0^k, u_1^k : h_{u_1^k}, u_2^k))$ and $(v_k) = ((v_0^k, v_1^k : h_{v_1^k}, v_2^k))$ be sequences of fuzzy numbers defined by $((\frac{k}{k+1}, \frac{3}{2} : 1, \frac{3k+2}{k+1}))$ and $((-\frac{3k+2}{k+1}, -\frac{3}{2} : 1, -\frac{k}{k+1}))$, respectively. (Caution: $v_k = (-1)u_k$). As similar to (16), we see that $T_e^C(v^k) = \frac{4}{3} \lim_n \frac{1}{n+1} \sum_{k=0}^{n} c_k$. Since $((u^k + v^k)) = ((-2, 0, 2))$, it is obtained that

$$T_e^C(u^k + v^k) = \lim_n \frac{1}{n+1} \sum_{k=0}^{n} \int_{x \in \mathbb{R}} h((u^k + v^k)(x))dx = \frac{8}{3} \lim_n \frac{1}{n+1} \sum_{k=0}^{n} c_k. \tag{17}$$

When does the equality $T_e^C(u^k + v^k) = T_e^C(u^k + (-1)u^k) = T_e^C(u^k) + T_e^C((-1)u^k)$ valid for sequences of fuzzy numbers? The answer for this question have been given in the following theorem:

Theorem 2.7 *Let us suppose that $u = (u^k)$ and $v = (v^k)$ be any sequences of fuzzy numbers and the entropy function h be linear. Then the inequality $T_e^C((u^k) + (v^k)) \geqslant T_e^C(u^k) + T_e^C(v^k)$ holds.*

Proof

$$
\begin{aligned}
T_e^C((u^k)+(v^k)) &= \lim_k \frac{1}{k+1}\sum_{i=0}^{k} \ell(u^i+u^i)(2\max\{h_{u_1^i},h_{v_1^i}\}-\frac{4}{3}(\max\{h_{u_1^i},h_{v_1^i}\})^2)\\
&\geqslant \lim_k \frac{1}{k+1}\sum_{i=0}^{k} \ell(u^i)(2h_{u_1^i}-\frac{4}{3}h_{u_1^i}^2)+\lim_k \frac{1}{k+1}\sum_{i=0}^{k} \ell(v^i)(2h_{v_1^i}-\frac{4}{3}h_{v_1^i}^2)\\
&= T_e^C((u^k))+T_e^C((v^k)). \qquad (18)
\end{aligned}
$$

If the sequences $u=(u^k)$ and $v=(v^k)$ are sequences of real numbers the equality holds. ■

Similar to Theorem 2.7, we can show that $T_e^{C^1}(nu^k)=|n|T_e^{C^1}(u^k)$, where n is an integer number.

THEOREM 2.8 *Let $\lambda(\mathscr{F})$ be any subset of the set $w(\mathscr{F})$ and $p_k(x)=c$. If for all $u=(u_k)\in\lambda(\mathscr{F})$ the $\lim_k \ell(u^k)$ exists then the $\lambda(\mathscr{F})$ either bounded or convergent sequence sets of sequences of fuzzy sets, an another condition does not exist.*

Proof Let us suppose that the $(u^k)\in\lambda(\mathscr{F})$ and $\lim_k \ell(u^k)$ exists. Since $0<h_{u_1^k}\leqslant 1$ the series

$$T_e^A(u^k)=c\lim_k \sum_{i=0}^{k}(2h_{u_1^k}-\frac{4}{3}h_{u_1^k}^2)\ell(u^k)$$

converges, thus we realize that the sequences $(\ell(u^k)2h_{u_1^k})$ and $(\ell(u^k)\frac{4}{3}h_{u_1^k}^2)$ of real numbers are convergent. For every k, using to mid point of the support (shortly, $\overline{supp(u^k)}$) and end point closuring of supports of the fuzzy sets $((u^k))$ we can construct a sequence membership functions as follows:

$$
u^k(x)=\begin{cases}
\frac{2h_{u_1^k}(x-\min\overline{supp(u^k)})}{\ell(u^k)-2\min\overline{supp(u^k)}}, \ if \ \ x\in[\min\overline{supp(u^k)},\frac{\ell(u^k)}{2}]\\
h_{u_1^k}-\frac{h_{u_1^k}(2x-\ell(u^k))}{2\max\overline{supp(u^k)}-\ell(u^k)}, \ if \ \ x\in(\frac{\ell(u^k)}{2},\max\overline{supp(u^k)})]\\
0, \ otherwise
\end{cases}, \quad k\in\mathbb{N}. \qquad (19)
$$

Clearly the sequence $(u^k(x))$ is convergent to a fuzzy set. ■

THEOREM 2.9 *Let $0<p_k(x)=c\leqslant 1$ for all $k\in\mathbb{N}$ and the sequence $((u^k))$ of fuzzy numbers be convergent to fuzzy number u^0. Then the $T_e^A(u^k)$ is not equal $e(u^0)$, in generally.*

Proof Let us consider the sequence $(u^k)=((-(k+2)(2k)^{-1},0:1,(5k+2)(2k)^{-1}))$ of fuzzy sets. Limit of sequence (u^k) is fuzzy set(fuzzy number) $(-\frac{1}{2},0:1,\frac{1}{2})$ and the entropy of this limit point is $e(u^0)=\frac{2c}{3}$ with $0<c\leqslant 1$ but $T_e^A(u^k)=\infty$. This shows to us $e(u_0)\neq T_e^A(u_k)$, in generally. We also easily see that, this theorem is valid for $T_e^{C^1}(u^k)$. ■

THEOREM 2.10 *Suppose that sequence $((u_k))$ and $((v_k))$ be sequences of fuzzy sets, $u_k\neq v_k$, $h_{u_1^k}=h_{v_1^k}$ for all $k\in\mathbb{N}$, but $|u_1^k-u_0^k|=|v_1^k-v_0^k|$ and $|u_2^k-u_1^k|=|v_2^k-v_1^k|$. Then the $T_e^A(u_k)$ is equal to $T_e^A(v_k)$.*

Proof The proof is clear from the equality (12) so we omit it. ■

In next section, we will investigate entropy of the electrocardiogram and give some comments. We know that, an electrocardiogram is an important test for any relevant heart diasases and the shortest way of identifying heart problems and you can detects cardiac (heart) abnormalities, as an example heart attacks, an enlarged heard or abnormal heart rhythms may cause heart failure, abnormal position of heart can be given., by measuring the electrical activity generated by the heart as it contacts, (for more, see [2]).

3. The Applications to ECG's of the Idea Entropy and Some Comments

It is a fact that, the long time can be spent for interpreting electrocardiographs results by cardiologists and sometimes small but important details can be unnoticed because of complexity of the ECG. The same situation is also valid for computerized electrocardiography. According to us, numerical values for ECG outputs be more reliable for cardiologists for interpreting ECG results. Furthermore, if the outputs are numerical then the consultation may be easy than consultation of the ECG papers. In this section we have proposed a new consultation method for cardiac problems which will be based upon numerical value of ECGs, (for ECG, see [4, 11, 27–29, 31]), see other applications of fuzzy sets to medicine [5], [6], [12], [17].

Quite simply every heart beats can be considered as therm of a sequence. Using to the waves P, QRS complex and T, we can construct the waves sequence $((P_k, (QRS)_k), T_k))$, where k is beat number or number of measurements and is finite. The graphical shapes of the waves P, QRS complex and T can imagine a membership functions a fuzzy set. With this idea, we can appoint an entropy value using to these membership functions which will be described below.

The entropy of the sequence $((P_k, (QRS)_k), T_k))$ can compute for finite or infinite many k and this computation gives to us a numerical value, not graphical. From numerical value, we can determine some cardiac problems. Namely, the sequence $((P_k, (QRS)_k), T_k))$ can divide three part for calculate entropy as follows:

(1) The entropy of the sequence (P_k) waves,
(2) The entropy of the sequence $((QRS)_k))$ complexes,
(3) The entropy of the sequence (T_k) waves.

In this case, we can assume that the total entropy of the heart is equal to

$$\mathscr{E} = e(P_k) + e((QRS)_k) + e(T_k). \tag{20}$$

Now we will summarize some information about electrocardiographs without deepening the subject.

The electrocardiograph records the electrical activity of the heart muscle and displays this data as a trace on a screen or on paper and, later, this data is interpreted by a medical practitioner. ECG's from healthy hearts have a characteristic shape. Any irregular in the heart rhythm or damage to the heart muscle can change the electrical activity of heart so that the shape of ECG is changed. Using this changes, we can investigate entropy of the heart rhythm or damage entropy of the heart muscle. It is known that, the QRS complex reflect the rapid depolarization of the right and left ventricles. The ventricles have a large muscle mass compared to the atria so the QRS complex usually has a much larger amplitude than the P- wave.

Furthermore, the heart movements are kept in check by various charges and pulses that change slightly on exertion, blood chemistry and strain. According to us, residence of skin and conductivity of blood are important for ECG, too. The conductivity and residence of the skin vary according to some minerals in the blood plasma such as calcium, chloride, potassium or glucose concentration in a diabetic patients blood. So we have to consider the conductivity of blood in the calculations of transmitting electric current and therefore in the entropy calculations for a heart. For blood conductivity properties, you can read to [10].

3.1 *The Entropy of The Waves Sequence (P_k) in Lead II and Some Comments*

Primary wave of a heart in ECG, is called P wave and shortly denoted with P, have an entropy value and it can be computed as follows:

$$e(P) = \int_{x\in\mathbb{R}} h_1(P(x))r(x)dx, \tag{21}$$

where the function $P(x)$ is membership function of the fuzzy $\mathscr{P}$ set that we will correspond to wave P and the function $r(x)$ is conductivity function (generally the function r is fix) of the body.

Experimental measurements showed to us, the wave P has maximal height about $2.5mm$ (or $0.3mV$), duration is shorter than 0.12 seconds (or 3 small square on the ECG paper), upright and rounded in lead II [4], [11]. In this section, all entropy calculations will be performed for ECG signals which come from lead II.

Using the maximal height of P wave as 0.3, the membership function $P(x)$ of the fuzzy $\mathscr{P}$ set which is correspond to wave P can write as follows:

$$P(x) = \begin{cases} 5x, \, x\in[0,0.06] \\ 0.6-5x, \, x\in(0.06,0.12] \\ 0, \text{ otherwise} \end{cases}. \tag{22}$$

It is clear that the support of the fuzzy set $\mathscr{P}$ is duration of the wave P and height is maximum height of wave P.

Let us take $supp\mathscr{P} \approx]0,0.12[$ and closure of the $supp\mathscr{P}$ be $\overline{supp\mathscr{P}} = [0,0.12]$ where the notation $supp\mathscr{P}$ denotes support of the $\mathscr{P}$.

In this case, we see that $h_1(P(x)) = \begin{cases} 20x-100x^2, \, x\in[0,0.06] \\ 0.96+7x-100x^2, \, x\in(0.06,0.12] \\ 0, \text{ otherwise} \end{cases}$. If we choose $r(x) = c$ in (21) then the the entropy of wave P is equal to

$$e(P) = 576\times 10^{-4}c \tag{23}$$

for normal wave P.

Let sequence $(P_k) = ((0, a_1^k : h_{a_1^k}, a_2^k))$ be finite sequence of the waves P, where the 0 denotes initial place of the wave P. The membership functions of sequence (P_k) are given as follows:

$$P_k(x) = \begin{cases} \frac{h_{a_1^k}}{a_1^k}x, \, x\in[0,a_1^k) \\ h_{a_1^k} - \frac{h_{a_1^k}}{(a_2^k-a_1^k)}(x-a_1^k), \, x\in[a_1^k,a_2^k] \\ 0, \text{ otherwise} \end{cases}, \quad k=1,2,...,n \tag{24}$$

where $0 < a_k^2 \leqslant 0.12$ is formation time of the k^{th} waves P_k as support of the fuzzy set which correspond to wave P, $a_k^1 = \frac{a_2^k}{2}$ and $0 < h_{a_1^k} \leqslant 0.3$ is height of the waves P at the k^{th} place. Using to equality (13) and membership functions of the wave

Table 1. Non-clinical P waves data

Gender: Male	Age:xx	Weight:xx	Height:xx							
Days	1	2	3	4	5	6	7	8	9	10
$m(h_{a_k^1})$	0.23	0.24	0.22	0.2	0.23	0.23	0.21	0.21	0.19	0.15
$m(a_2^k)$	0.11	0.09	0.05	0.07	0.09	0.11	0.10	0.09	0.11	0.11
$e(P_k)^{\text{a}}$	0,0428	0,0362	0,0187	0,0242	0,0350	0,04284	0,0361	0,0325	0,0365	0,0297
$S(P_k, P)^{\text{a}}$	0,7628	0,788	0,7076	0,65	0,7551	0,7628	0,693	0,6895	0,6301	0,4975

[a]This values are approximate values of $e(P_k)$ and $S(P_k, P)$.

P which is given in (24), we can give the total Cesàro entropy definition for the sequences of waves P_k as follows:

Definition 3.1 *Let (P_k) be finite sequence of the waves P and the membership functions of (P_k) be as in the (24). Then total Cesàro entropy of the sequence (P_k) is*

$$T_e^{C^1}(P_k) = \frac{1}{k+1}\sum_{i=0}^{k} c_i a_2^i (2h_{a_1^i} - \frac{4}{3}h_{a_1^i}^2) S(P_k, P), \tag{25}$$

where c_i is resistance of the dry skin in the i. sample, k is number of sample of P wave and $S(P_k, P)$ is similarity degree between of the waves P_k and P.

Let the resistance of the dry skin be fix that is if c_i equal to c at the each every i. place then the (25) is turn to

$$T_e^{C^1}(P_k) = \frac{c}{k+1}\sum_{i=0}^{k} a_2^i (2h_{a_1^i} - \frac{4}{3}h_{a_1^i}^2) S(P_k, P). \tag{26}$$

Example 3.2 Let us suppose, the wave P values as height and width as given in Table 1 for 10 measurements with fix conductivity of blood and residence of the skin. Note that these data are not clinical measures. In this mean, the sequence (P_k) is in the set $\varphi(\mathscr{F})$.

The notations $m(h_{a_k^1})$ and $m(a_2^k)$ in Table 1 denotes measured height and durations of the wave P in day. Then from (26), we see that the Cesàro total entropy of the wave P is

$$T_e^{C^1}(P_k) = 23,472 \times 10^{-4}c \tag{27}$$

for 10 beats. If we compare (23) and (27), the P wave properties of the hearth which given above example is very low than normal value. Using to (7), we can give a graphic (see, Figure 2) for 10 sample of P wave which given in the Table 1 as follows:

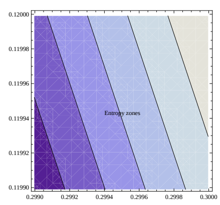

Figure 1.
Graphical representation of $e(P_k)$ of the normal P wave.

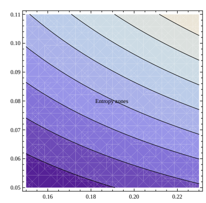

Figure 2.
Graphical representation of $e(P_k)$ for Table 1 values.

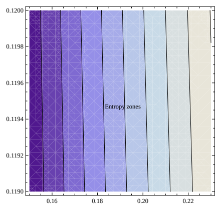

Figure 3.
The values $h_{a_k^2}$ nearly fix but values a_2^k variable.

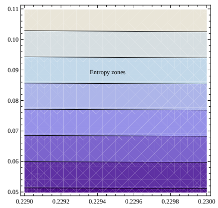

Figure 4.
The values a_2^k nearly fix but the values $h_{a_k^2}$ variable.

The Figure 1 is called entropy graphic for the normal wave P. If we compare the Figures 1 and 2 then we see that the height and duration of the P wave when changed with any effect, the all entropy zones are curl to upward. It can be consider that the magnitude of the curl is P wave degenerations.

If the $h_{a_k^1}$ is fix but the value a_2^k be variable and conversely the $h_{a_k^1}$ is variable but the value a_2^k be fix then graphical representation of the entropy zones showed in Figure 3 and Figure 4, respectively.

As similar to (25), the A- entropy of the sequence wave P is

$$T_e^A(P_k) = 2347,21 \times 10^{-4} c \tag{28}$$

from (13). But normal A-entropy value for 10 beats should be $5760 \times 10^{-4}c$ and the P wave value in (28) very low than $5760 \times 10^{-4}c$, where c is resistance of the dry skin in the i^{th} time.

From Example 3.2 and explanations is that given above, we give an important definition as follows:

Definition 3.3 *Let $\epsilon > 0$ be very small positive real number and $n \in \mathbb{N}$. If*

$$|e(P) - e(P_k)| < \epsilon, \quad k = 1, 2, ..., n \tag{29}$$

for every finite sequence of the wave P then the P wave properties of the heart is normal.

Comment 1.

From Definition 3.3, and the (27), we can determine that the wave P properties which was given in the Example 3.4 is not normal.

Comment 2.

We know that the value of the $S(P_k, P)$ must be $0 \leqslant S(P_k, P) \leqslant 1$ for every $k \in \mathbb{N}$. After a certain place, if P_k waves do not exists, or the similarity values $S(P_k, P)$ nearly to the zero then the entropy of atrial depolarization of the heard, the $T_e^A(P_k)$ is near to zero. In this case we can say that this is a risk (for example, it can indicate hyperkalemia or hypokalemia or right atrial enlargement [30]) for this heart in the future.

Comment 3.

If the values of the $T_e^A(P_k)$ less than $234,72 \times 10^{-4}c$ then, we can say that, there is a risk (for example, it can indicate hyperkalemia or hypokalemia or right atrial enlargement [30]) for this heard in the future.

3.2 *The Entropy of The Sequence* (T_k) *and Some Comments*

No widely accepted criteria exist regarding T wave amplitude. As a general rule, T wave amplitude corresponds with the amplitude of the preceding R wave, though the tallest T waves are seen in leads V3 and V4 [26]. But many medical sources claim that the duration of the T wave is 0.10 to 0.25 seconds or greater and the amplitude of the T wave is less than 5 mm (or 0.5mV) [4], [11]. Therefore we will calculate two different entropy (T_1 and T_2) for wave T after we will use arithmetic mean of the two different entropy values in our comparisons.

Firstly, if we consider equality (6) then entropy of the wave T, for duration 0.10 and amplitude 0.5 with membership functions $T_1(x) = \begin{cases} 10x, \ x \in [0, 0.05] \\ 1 - 10x, \ x \in (0.05, 0.10] \\ 0, \text{ otherwise} \end{cases}$, $T_2(x) = \begin{cases} 4x, \ x \in [0, 0.125] \\ 1 - 4x, \ x \in (0.125, 0.25] \\ 0, \text{ otherwise} \end{cases}$ and $r(x) = c$ fix, then

$$e(T_1) = 66.6667 \times 10^{-4}c \tag{30}$$

and secondly, for duration 0.25 and amplitude 0.5 the $e(T)$ is

$$e(T_2) = 166.667 \times 10^{-4}c, \tag{31}$$

where $h(x) = h_1(x)$. Therefore, it can be chosen
for entropy of the wave T as reference interval

$$66.6667 \times 10^{-4}c \leqslant e(T) \leqslant 166.667 \times 10^{-4}c \tag{32}$$

for diagnosis of some cardiac wave T problems. Probably, the best value for

$$e(T) = 116,66685 \times 10^{-4}c$$

as arithmetics mean of the $e(T_1)$ and $e(T_2)$. If we compare the Figures 5 and 6 then

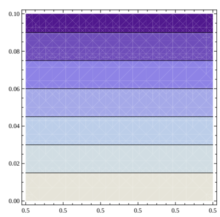

Figure 5.
Graphical representation of $e(T)$of the Twave for duration 0.10 and amplitude 0.5.

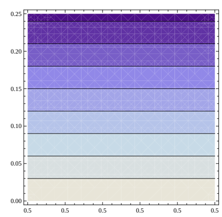

Figure 6.
Graphical representation of $e(T)$ of the Twave for duration 0.25 and amplitude 0.5.

we see that entropy graphics for two different values are almost same. That is to say normal wave T entropy graphic should be as in Figure 5 or Figure 6.

Table 2. Non-clinical data for waves T

Gender: Male	Age:xx	Weight:xx	Height:xx							
Days	1	2	3	4	5	6	7	8	9	10
$m(h_{a_k^1})$	0.43	0.42	0.40	0.49	0.47	0.47	0.45	0.48	0.49	0.50
$m(a_2^k)$	0.22	0.24	0.25	0.25	0.20	0.21	0.19	0.17	0.11	0.1
$e(T_k)^{\mathrm{b}}$	0,1349	0,1451	0,1466	0,1649	0,1387	0,1355	0,1197	0,1109	0,0725	0,0733
$S(T_k, T)^{\mathrm{b}}$	0,6627	0,6714	0,7012	0,5724	0,6127	0,6095	0,6433	0,6093	0,6091	0,597

[b]This values are approximate values of $e(P_k)$ and $S(P_k, P)$.

If the duration is very short and the amplitude is 0.5 then the peak of the wave T is sharped. If, also the duration is 0.10 (or 0.25) and the amplitude is very low (to say that 0.1) then the peak of the wave T is flattened. Under these conditions, the entropy value and graphics of this wave T showed in Figure 7 and Figure 8.

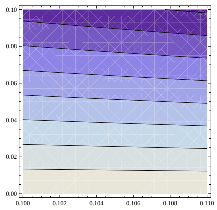

Figure 7.
Graphical representation of $e(T)$ of the T wave for duration 0.10 and amplitude low (0.1-0.11).

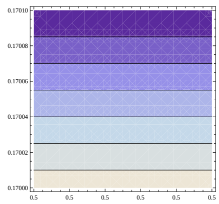

Figure 8.
Graphical representation of $e(T)$ of the T wave for duration very low (0.0001) and amplitude 0.5.

Example 3.4 Let us suppose, the wave T values as height and width as given in Table 2 for 10 measurements with fix conductivity of body. Note that these data are not clinical measures. In this mean, the sequence $((T_k))$ is in the set $\varphi(\mathscr{F})$.

The notations $m(h_{a_k^1})$ and $m(a_2^k)$ in Table 2 denotes measured height and durations of the wave T in day, respectively. Then from (26), we see that

$$T_e^A(T_k) = 77,8642358 \times 10^{-4}c \tag{33}$$

for 10 beats. If we compare (32) and (33), the T wave properties of the hearth which given above example is normal. Using to (7), we can give a graphic (see, Figure 9) for 10 value which given in the Table 2. The graphic is called entropy

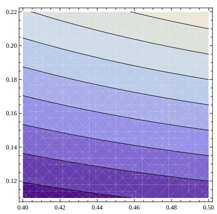

Figure 9.
Graphical representation of $e(T)$ of the wave T for Table 2.

graphic for the wave T which shown in the Figure 2. If we compare the Figures 1 and 2 then we see that the height and duration of the T wave when changed with any effect, the all entropy zones are curl to upward. It can be consider that the magnitude of the curl is T wave degenerations.

Comment 4.

The graphical representation of the normal wave T must be similar to Figures 5 and 6 for upper bound and lower bound of wave T. Otherwise, for example, it is similar to Figure 9 then we have to think that there is a cardiac problem.

Comment 5.

The inequality $66.6667 \times 10^{-4}c \leqslant e(T) \leqslant 166.667 \times 10^{-4}c$ must be satisfy for the wave T, otherwise, again we should think that there is a deformation for wave P.

4. Conclusions and Suggestions

The conclusions can be summarized as follows:

(1) The entropy of the wave P for normal heart should be $61.056 \times 10^{-4}c$.
(2) The graphical representation of the normal wave P should similar to Figure 1.
(3) If the duration is fix but height is being altered by any reason then lines in graphical representation of the wave P becomes steeper.
(4) The lines in the graphical representation of the wave T should be almost parallel to horizontal axis.

As a suggestion, clearly, one can define entropy value and graphical representations of QRS complex to similar entropy value wave P or wave T. So any numerical value can obtain for (20). If entropy value of the QRS complex is calculate then we can give a numerical entropy value for (20).

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