

## Jacobi Operational Matrix Approach for Solving Systems of Linear and Nonlinear Integro–Differential Equations

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**Abstract.** This paper aims to construct a general formulation for the shifted Jacobi operational matrices of integration and product. The main aim is to generalize the Jacobi integral and product operational matrices to the solving system of Fredholm and Volterra integro–differential equations which appear in various fields of science such as physics and engineering. The Operational matrices together with the collocation method are applied to reduce the solution of these problems to the solution of a system of algebraic equations. Indeed, to solve the system of integro–differential equations, a fast algorithm is used for simplifying the problem under study. The method is applied to solve system of linear and nonlinear Fredholm and Volterra integro–differential equations. Illustrative examples are included to demonstrate the validity and efficiency of the presented method. It is further found that the absolute errors are almost constant in the studied interval. Also, several theorems related to the convergence of the proposed method, will be presented.

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## 1. Introduction

Finding the analytical solutions of functional equations has been devoted the attention of mathematicians' interest in recent years. Several methods are proposed to

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achieve this purpose, such as [3, 5–12, 16–19, 26, 29, 30, 33, 34]. Systems of integro-differential equations arise in the mathematical modeling of many phenomena. Various techniques have been used for solving these systems such as, Adomian decomposition method [13], Homotopy perturbation method [2, 14], Variational iteration method [15], the Tau method [1, 32], differential transform method [4], and others. Between of present methods, spectral methods have been used to solve different functional equations, because of their high accuracy and easy applying. Specific types of spectral methods that more applicable and widely used, are the Galerkin, collocation, and Tau methods [20–25, 27, 31]. The importance of Sturm–Liouville problems for spectral methods lies in the fact that the spectral approximation of the solution of a functional equation is usually regarded as a finite expansion of eigenfunctions of a suitable Sturm–Liouville problem. In recent years, many different orthogonal functions and polynomials have been used to approximate the solution of various functional equations. The main goal of using orthogonal basis is that the equation under study reduces to a system of linear or nonlinear algebraic equations. This can be done by truncating series of functions with orthogonal basis for the solution of equations and using the operational matrices. In this paper, Jacobi polynomials, on interval  $[0, 1]$ , have been considered for solving systems of integro-differential equations. The Jacobi polynomials  $P_i^{(\alpha, \beta)}(x)$  ( $i \geq 0, \alpha, \beta > -1$ ) play important roles in mathematical analysis and its applications [20]. It is proven that Jacobi polynomials are precisely the only polynomials arising as eigenfunctions of a singular Sturm–Liouville problem [22, 31]. This class of polynomials comprises all the polynomial solution to singular Sturm–Liouville problems on  $[-1, 1]$ . Chebyshev, Legendre, and ultraspherical polynomials are particular cases of the Jacobi polynomials.

In this paper, the shifted Jacobi operational matrices of integration and product is introduced, which is based on Jacobi collocation method for solving numerically the systems of the linear and nonlinear Fredholm and Volterra integro-differential equations on the interval  $[0, 1]$ , to find the approximate solutions  $(u_i)_N(x)$ ,  $i = 1, \dots, n$ . The resultant systems are collocated at  $n(N + 1)$  nodes of the shifted Jacobi–Gauss interpolation on the interval  $(0, 1)$ . These equations generate  $n(N + 1)$  linear or nonlinear algebraic equations. The nonlinear systems can be solved using Newton iterative method.

The remainder of this paper is organized as follows: The Jacobi polynomials and some their properties are introduced in Section 2. In Section 3, The Jacobi operational matrices of integration and product are derived. In Section 4, the convergence of the method is studied. Section 5 is devoted to applying the Jacobi operational matrices for solving system of integro-differential equations. In Section 6, the proposed method is applied to several examples. A conclusion is presented in Section 7.

## 2. Jacobi Polynomials and Their Properties

The Jacobi polynomials, associated with the real parameters  $(\alpha, \beta > -1)$ , are a sequence of polynomials  $P_i^{(\alpha, \beta)}(t)$  ( $i = 0, 1, 2, \dots$ ), each of degree  $i$ , are orthogonal with Jacobi weighted function  $w(x) = (1 - x)^\alpha(1 + x)^\beta$  over  $I = [-1, 1]$ , and with the following orthogonality condition:

$$\int_{-1}^1 P_n^{(\alpha, \beta)}(t) P_m^{(\alpha, \beta)}(t) w(t) dt = h_n \delta_{nm},$$

where  $\delta_{nm}$  is the Delta function and

$$h_n = \frac{2^{\alpha+\beta+1}\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)}{(2n + \alpha + \beta + 1)n! \Gamma(n + \alpha + \beta + 1)}.$$

These polynomials can be generated with the following recurrence formula:

$$P_i^{(\alpha,\beta)}(t) = \frac{(\alpha + \beta + 2i - 1) \left( \alpha^2 - \beta^2 + t(\alpha + \beta + 2i)(\alpha + \beta + 2i - 2) \right)}{2i(\alpha + \beta + i)(\alpha + \beta + 2i - 2)} P_{i-1}^{(\alpha,\beta)}(t) - \frac{(\alpha + i - 1)(\beta + i - 1)(\alpha + \beta + 2i)}{i(\alpha + \beta + i)(\alpha + \beta + 2i - 2)} P_{i-2}^{(\alpha,\beta)}(t), \quad i = 2, 3, \dots,$$

where

$$P_0^{(\alpha,\beta)}(t) = 1, \quad P_1^{(\alpha,\beta)}(t) = \frac{\alpha + \beta + 2}{2}t + \frac{\alpha - \beta}{2}.$$

In order to use these polynomials on the interval  $[0, 1]$ , shifted Jacobi polynomials are defined by introducing the change of variable  $t = 2x - 1$ . Let the shifted Jacobi polynomials  $P_i^{(\alpha,\beta)}(2x - 1)$  be denoted by  $R_i^{(\alpha,\beta)}(x)$ , then  $R_i^{(\alpha,\beta)}(x)$  can be generated from following formula:

$$R_i^{(\alpha,\beta)}(x) = \frac{(\alpha + \beta + 2i - 1) \left( \alpha^2 - \beta^2 + (2x - 1)(\alpha + \beta + 2i)(\alpha + \beta + 2i - 2) \right)}{2i(\alpha + \beta + i)(\alpha + \beta + 2i - 2)} \times R_{i-1}^{(\alpha,\beta)}(x) - \frac{(\alpha + i - 1)(\beta + i - 1)(\alpha + \beta + 2i)}{i(\alpha + \beta + i)(\alpha + \beta + 2i - 2)} R_{i-2}^{(\alpha,\beta)}(x), \quad i = 2, 3, \dots, \tag{1}$$

where

$$R_0^{(\alpha,\beta)}(x) = 1, \quad R_1^{(\alpha,\beta)}(x) = \frac{\alpha + \beta + 2}{2}(2x - 1) + \frac{\alpha - \beta}{2}.$$

*Remark 1* Of this polynomials, the most commonly used are the shifted Gegenbauer polynomials,  $C_{S,i}^\alpha(x)$ , the shifted Chebyshev polynomials of the first kind,  $T_{S,i}(x)$ , the shifted Legendre polynomials,  $P_{S,i}(x)$ , the shifted Chebyshev polynomials of the second kind,  $U_{S,i}(x)$ . These orthogonal polynomials are related to the shifted Jacobi polynomials by the following relations.

$$C_{S,i}^\alpha(x) = \frac{i!\Gamma(\alpha + \frac{1}{2})}{\Gamma(i + \alpha + \frac{1}{2})} R_i^{(\alpha-\frac{1}{2},\beta-\frac{1}{2})}(x), \quad T_{S,i}(x) = \frac{i!\Gamma(\frac{1}{2})}{\Gamma(i + \frac{1}{2})} R_i^{(-\frac{1}{2},-\frac{1}{2})}(x),$$

$$P_{S,i}(x) = R_i^{(0,0)}(x), \quad U_{S,i}(x) = \frac{(i + 1)!\Gamma(\frac{1}{2})}{\Gamma(i + \frac{3}{2})} R_i^{(\frac{1}{2},\frac{1}{2})}(x).$$

The analytic form of the shifted Jacobi polynomials,  $R_i^{(\alpha,\beta)}(x)$ , is given by:

$$R_i^{(\alpha,\beta)}(x) = \sum_{k=0}^i \frac{(-1)^{(i-k)}\Gamma(i + \beta + 1)\Gamma(i + k + \alpha + \beta + 1)x^k}{\Gamma(k + \beta + 1)\Gamma(i + \alpha + \beta + 1) (i - k)! k!}, \tag{2}$$

Some properties of the shifted Jacobi polynomials are presented as:

$$\begin{aligned} (1) \quad R_i^{(\alpha,\beta)}(0) &= (-1)^i \binom{i+\alpha}{i}, \\ (2) \quad R_i^{(\alpha,\beta)}(1) &= (-1)^i \binom{i+\beta}{i}, \\ (3) \quad \frac{d^i}{dx^i} R_n^{(\alpha,\beta)}(x) &= \frac{\Gamma(n+\alpha+\beta+i+1)}{\Gamma(n+\alpha+\beta+1)} R_{n-i}^{(\alpha+i,\beta+i)}(x). \end{aligned}$$

The orthogonality condition of shifted Jacobi polynomials is:

$$\int_0^L R_j^{(\alpha,\beta)}(x) R_k^{(\alpha,\beta)}(x) W^{(\alpha,\beta)}(x) dx = \theta_k \delta_{jk}, \quad (3)$$

where  $W^{(\alpha,\beta)}(x)$ , the shifted weighted function, and  $\theta_k$  are as follows:

$$W^{(\alpha,\beta)}(x) = x^\beta (1-x)^\alpha, \quad \theta_k = \frac{h_k}{2^{\alpha+\beta+1}}.$$

LEMMA 2.1 *The shifted Jacobi polynomial  $R_n^{(\alpha,\beta)}(x)$  can be obtained in the form of:*

$$R_n^{(\alpha,\beta)}(x) = \sum_{i=0}^n p_i^{(n)} x^i,$$

where  $p_i^{(n)}$  are,

$$p_i^{(n)} = (-1)^{n-i} \binom{n+\alpha+\beta+i}{i} \binom{n+\alpha}{n-i}.$$

*Proof*  $p_i^{(n)}$  can be obtained as:

$$p_i^{(n)} = \frac{1}{i!} \frac{d^i}{dx^i} R_n^{(\alpha,\beta)}(x) \Big|_{x=0}.$$

Now, by using properties (1) and (3) in above, the lemma can be proved. ■

LEMMA 2.2 *For  $m > 0$ , one has:*

$$\int_0^1 x^m R_j^{(\alpha,\beta)}(x) W^{(\alpha,\beta)}(x) dx = \sum_{l=0}^j p_l^{(j)} B(m+l+\beta+1, \alpha+1),$$

where  $B(s, t)$  is the Beta function and is defined as:

$$B(s, t) = \int_0^1 v^{s-1} (1-v)^{t-1} dv = \frac{\Gamma(s) \Gamma(t)}{\Gamma(s+t)}.$$

*Proof* Using Lemma 2.1 and  $W^{(\alpha,\beta)} = (1 - x)^\alpha x^\beta$  one has:

$$\begin{aligned} \int_0^1 x^m R_j^{(\alpha,\beta)}(x) W(\alpha,\beta)(x) dx &= \sum_{l=0}^j p_l^{(j)} \int_0^1 x^m x^l (1-x)^\alpha x^\beta dx \\ &= \sum_{l=0}^j p_l^{(j)} \int_0^1 (1-x)^\alpha x^{m+l+\beta} dx \\ &= \sum_{l=0}^j p_l^{(j)} B(m+l+\beta+1, \alpha+1). \end{aligned}$$

■

### 3. Convergence Analysis

In this section, some theorems on convergence analysis and error estimation of the proposed method are provided.

Let  $\Omega = (0, 1)$  and for  $r \in \mathbb{N}$  ( $\mathbb{N}$  is the set of all non-negative integers), the weighted Sobolov space  $H_{W^{(\alpha,\beta)}}^r(\Omega)$  is defined in the usual way and is denoted inner product, semi-norm and norm by  $(u, v)_{r,W^{(\alpha,\beta)}}$ ,  $|v|_{r,W^{(\alpha,\beta)}}$  and  $\|v\|_{r,W^{(\alpha,\beta)}}$ , respectively. In particular,

$$L_{W^{(\alpha,\beta)}}^2(\Omega) = H_{W^{(\alpha,\beta)}}^0(\Omega), \quad (u, v)_{W^{(\alpha,\beta)}} = (u, v)_{0,W^{(\alpha,\beta)}},$$

and

$$\begin{aligned} \|v\|_{W^{(\alpha,\beta)}} &= \|v\|_{0,W^{(\alpha,\beta)}}, \\ H_{W^{(\alpha,\beta)}}^r(\Omega) &= \{f \mid f \text{ is measurable and } \|v\|_{r,W^{(\alpha,\beta)}} < \infty\}, \\ \|u\|_{r,W^{(\alpha,\beta)}}^2 &= \sum_{k=0}^r \|\partial_x^k u\|_{W^{(\alpha+k,\beta+k)}}^2, \\ |u|_{r,W^{(\alpha,\beta)}} &= \|\partial_x^r u\|_{W^{(\alpha+r,\beta+r)}}. \end{aligned}$$

A function  $u(x) \in H_{W^{(\alpha,\beta)}}^r(\Omega)$  can be expanded in  $\mathbf{P}^{(N,\alpha,\beta)} = \text{span}\{R_0^{(\alpha,\beta)}(x), R_1^{(\alpha,\beta)}(x), \dots, R_N^{(\alpha,\beta)}(x)\}$  as the below formula:

$$u(x) = \sum_{j=0}^{\infty} c_j R_j^{(\alpha,\beta)}(x), \tag{4}$$

where the coefficients  $c_j$  are given by:

$$c_j = \frac{1}{\theta_j} \int_0^1 R_j^{(\alpha,\beta)}(x) u(x) W^{(\alpha,\beta)}(x) dx, \quad j = 0, 1, 2, \dots \tag{5}$$

By noting in practice, only the first  $(n + 1)$ -terms shifted polynomials are consid-

ered, then one has:

$$u(x) \simeq u_N(x) = \sum_{j=0}^N c_j R_j^{(\alpha, \beta)}(x) = \Phi^T(x) C, \quad (6)$$

where

$$C = [c_0, c_1, \dots, c_N]^T, \quad \Phi(x) = [R_0^{(\alpha, \beta)}(x), R_1^{(\alpha, \beta)}(x), \dots, R_N^{(\alpha, \beta)}(x)]^T. \quad (7)$$

Since  $\mathbf{P}^{(N, \alpha, \beta)}$  is a finite dimensional vector space,  $u(x)$  has a unique best approximation from  $\mathbf{P}^{(N, \alpha, \beta)}$ , say  $u_N(x) \in \mathbf{P}^{(N, \alpha, \beta)}$ , that is:

$$\forall y \in \mathbf{P}^{(N, \alpha, \beta)}, \quad \|u(x) - u_N(x)\|_{W^{(\alpha, \beta)}} \leq \|u(x) - y\|_{W^{(\alpha, \beta)}}.$$

In [28] is shown that for any  $u(x) \in H_{W^{(\alpha, \beta)}}^r(\Omega)$ ,  $r \in \mathbb{N}$  and  $0 \leq \mu \leq r$ , a positive constant  $c$  independent of any function,  $N$ ,  $\alpha$ , and  $\beta$  exist that:

$$\|u(x) - u_N(x)\|_{\mu, W^{(\alpha, \beta)}} \leq c(N(N + \alpha + \beta))^{\frac{\mu-r}{2}} |u|_{r, W^{(\alpha, \beta)}}. \quad (8)$$

Let  $u(x)$  is  $N + 1$  times continuously differentiable. The following theorem can present an upper bound for estimating the error.

**THEOREM 3.1** *Let  $u(x) : [x_0, 1] \rightarrow \mathbb{R}$  is  $N + 1$  times continuously differentiable for  $x_0 > 0$ , and*

$$\mathbf{P}^{(N, \alpha, \beta)} = \text{span}\{R_0^{(\alpha, \beta)}(x), R_1^{(\alpha, \beta)}(x), \dots, R_N^{(\alpha, \beta)}(x)\}.$$

*If  $u_N = \Phi^T C$  is the best approximation to  $u(x)$  from  $\mathbf{P}^{(N, \alpha, \beta)}$  then the error bound is presented as follows:*

$$\|u(x) - u_N(x)\|_{W^{(\alpha, \beta)}} \leq \frac{MS^{N+1}}{(N+1)!} \sqrt{B(\alpha+1, \beta+1)},$$

where

$$M = \max_{x \in [x_0, 1]} u^{(N+1)}(x), \quad S = \max\{1 - x_0, x_0\}.$$

*Proof* Let consider the Taylor expansion of function  $u(x)$  namely  $\tilde{u}_N(x)$ . Therefore, one has:

$$|u(x) - \tilde{u}_N(x)| \leq \frac{(x - x_0)^{N+1}}{(N+1)!} u^{(N+1)}(\xi), \quad \xi \in (x_0, 1).$$

Since  $\Phi^T C$  is the best approximation to  $u(x)$  from  $\mathbf{P}^{(N, \alpha, \beta)}$ , and  $y(x) \in \mathbf{P}^{(N, \alpha, \beta)}$  one has:

$$\begin{aligned} \|u(x) - u_N(x)\|_{W^{(\alpha, \beta)}}^2 &\leq \|u(x) - \tilde{u}_N(x)\|_{W^{(\alpha, \beta)}}^2 \\ &\leq \frac{M^2}{((N+1)!)^2} \int_0^1 (x - x_0)^{2(N+1)} W^{(\alpha, \beta)}(x) dx. \end{aligned}$$

Since  $W^{(\alpha,\beta)}(x)$  is always positive in  $(0, 1)$ , choosing  $S = \max\{1 - x_0, x_0\}$  leads to:

$$\| u(x) - u_N(x) \|_{W^{(\alpha,\beta)}}^2 \leq \frac{M^2 S^{2(N+1)}}{((N + 1)!)^2} B(\alpha + 1, \beta + 1).$$

■

This error bound shows approximation of polynomials converges to  $u(x)$  as  $N \rightarrow \infty$ . The following theorem shows the coefficients,  $c_j$ , tend to zero as  $N$  increases.

**THEOREM 3.2** *Let  $u(x)$  be a function such that  $W^{(\alpha,\beta)}(x)u(x)$  is integrable over  $\Omega = (0, 1)$ , and let:*

$$u(x) = \sum_{j=0}^{\infty} c_j R_j^{(\alpha,\beta)}(x), \quad c_j = \int_0^1 R_j^{(\alpha,\beta)}(x) u(x) W^{(\alpha,\beta)}(x) dx, \quad (9)$$

then  $\lim_{j \rightarrow \infty} c_j = 0$ .

*Proof* Let  $s_N(x)$  be the partial sum of the series (9) through terms of  $N$ th degree

$$s_N(x) = \sum_{j=0}^N R_j^{(\alpha,\beta)}(x).$$

Then, the definition of the  $c_j$  's and the properties of the orthogonality of  $R_j^{(\alpha,\beta)}$ , leads to the following relation:

$$\int_0^1 s_N(x) u(x) W^{(\alpha,\beta)}(x) dx = \sum_{j=0}^N \int_0^1 u(x) R_j^{(\alpha,\beta)}(x) W^{(\alpha,\beta)}(x) dx = \sum_{j=0}^N \theta_j c_j^2.$$

If  $W^{(\alpha,\beta)}(x)u^2(x)$  as well as  $W^{(\alpha,\beta)}(x)u(x)$  is integrable, then one has:

$$\begin{aligned} \int_0^1 W^{(\alpha,\beta)}(x)[u(x) - s_N(x)]^2 dx &= \int_0^1 W^{(\alpha,\beta)}(x)[u(x)]^2 dx \\ &\quad - 2 \int_0^1 W^{(\alpha,\beta)}(x)u(x)s_N(x) dx \\ &\quad + \int_0^1 W^{(\alpha,\beta)}(x)[s_N(x)]^2 dx \\ &= \int_0^1 W^{(\alpha,\beta)}(x)[u(x)]^2 dx - \sum_{j=0}^N \theta_j c_j^2. \end{aligned}$$

Therefore,

$$\sum_{j=0}^N \theta_j c_j^2 \leq \int_0^1 W^{(\alpha,\beta)}(x)[u(x)]^2 dx.$$

Consequently,  $\sum_{j=0}^{\infty} \theta_j c_j^2$  is convergent and  $\lim_{j \rightarrow \infty} c_j = 0$ .

■

Theorem 3.2 shows that a good approximate solution can be obtained by means of the finite numbers of Jacobi polynomials in the series (9). Now, the theorem on convergence of the proposed method is provided.

**THEOREM 3.3** *The series solution Eq.(6) using Jacobi collocation method converges towards  $u(x)$  in Eq.(4).*

*Proof* Let  $L_{W^{(\alpha,\beta)}}^2(\Omega)$  be the Hilbert space and let  $u(x) = \sum_{j=0}^N c_j R_j^{(\alpha,\beta)}(x)$  where,

$$c_j = \frac{1}{\theta_j} \left( u(x), R_j^{(\alpha,\beta)}(x) \right)_{0,W^{(\alpha,\beta)}}.$$

Define the sequence of partial sums  $S_n$  as follows:

$$S_n(x) = \sum_{j=0}^n c_j R_j^{(\alpha,\beta)}(x).$$

Let  $S_n$  are  $S_m$  are arbitrary partial sums with  $n > m$ . It is going to prove that  $S_n$  is a Cauchy sequence in Hilbert space  $L_{W^{(\alpha,\beta)}}^2(\Omega)$ .

$$\begin{aligned} (u(x), S_n(x))_{0,W^{(\alpha,\beta)}} &= \left( u(x), \sum_{j=0}^n c_j R_j^{(\alpha,\beta)}(x) \right)_{0,W^{(\alpha,\beta)}} \\ &= \sum_{j=0}^n \bar{c}_j \left( u(x), R_j^{(\alpha,\beta)}(x) \right)_{0,W^{(\alpha,\beta)}} \\ &= \sum_{j=0}^n \theta_j \bar{c}_j c_j \\ &= \sum_{j=0}^n \theta_j |c_j|^2. \end{aligned}$$

So, one has:

$$\begin{aligned} \left\| \sum_{j=m+1}^n c_j R_j^{(\alpha,\beta)}(x) \right\|_{0,W^{(\alpha,\beta)}}^2 &= \left( \sum_{j=m+1}^n c_j R_j^{(\alpha,\beta)}(x), \sum_{i=m+1}^n c_i R_i^{(\alpha,\beta)}(x) \right)_{0,W^{(\alpha,\beta)}} \\ &= \sum_{j=m+1}^n \sum_{i=m+1}^n \bar{c}_j c_i \left( R_j^{(\alpha,\beta)}(x), R_i^{(\alpha,\beta)}(x) \right)_{0,W^{(\alpha,\beta)}} \\ &= \sum_{j=m+1}^n \theta_j |c_j|^2. \end{aligned}$$

That is,

$$\| S_n(x) - S_m(x) \|_{0,W^{(\alpha,\beta)}}^2 = \sum_{j=m+1}^n \theta_j |c_j|^2, \quad \text{for } n > m.$$

From Bessel inequality, one has  $\sum_{j=0}^{\infty} \theta_j |c_j|^2$  is convergent and hence  $\| S_n(x) - S_m(x) \|_{0,W^{(\alpha,\beta)}}^2 \rightarrow 0$  as  $m, n \rightarrow \infty$ . So,  $S_n$  is a Cauchy sequence and it converges



to say  $s$ . It is asserted that  $u(x) = s$ . So,

$$\begin{aligned} \left( s - u(x), R_j^{(\alpha,\beta)}(x) \right)_{0,W^{(\alpha,\beta)}} &= \left( s, R_j^{(\alpha,\beta)}(x) \right)_{0,W^{(\alpha,\beta)}} - \left( u(x), R_j^{(\alpha,\beta)}(x) \right)_{0,W^{(\alpha,\beta)}} \\ &= \left( \lim_{n \rightarrow \infty} S_n(x), R_j^{(\alpha,\beta)}(x) \right)_{0,W^{(\alpha,\beta)}} - \theta_j c_j \\ &= \lim_{n \rightarrow \infty} \left( S_n(x), R_j^{(\alpha,\beta)}(x) \right)_{0,W^{(\alpha,\beta)}} - \theta_j c_j \\ &= \theta_j c_j - \theta_j c_j \\ &\Rightarrow \left( s - u(x), R_j^{(\alpha,\beta)}(x) \right)_{0,W^{(\alpha,\beta)}} = 0. \end{aligned}$$

Hence,  $u(x) = s$  and  $\sum_{j=0}^n c_j R_j^{(\alpha,\beta)}(x)$  converges to  $u(x)$ . ■

#### 4. The Jacobi Operational Matrices

In performing arithmetic and other operations on the Jacobi basis, we frequently encounter the integration of the vector  $\Phi(x)$  defined in Eq.(7) and it is necessary to evaluate the product of  $\Phi(x)$  and  $\Phi^T(x)$ , which called the product matrix for the Jacobi polynomials basis. In this section, these operational matrices are derived.

##### 4.1 The Jacobi Operational Matrix of Integration

In this subsection, Jacobi operational matrix of the integration is derived. Let,

$$\int_0^x \Phi(t) dt \simeq P \Phi(x), \tag{10}$$

where matrix  $P_{(N+1) \times (N+1)}$  is called the Jacobi operational matrix of the integration. The elements of this matrix are obtained as follows:

**THEOREM 4.1** *Let  $P$  is  $(N + 1) \times (N + 1)$  operational matrix of integration. Then the elements of this matrix are obtained as:*

$$P_{ij} = \frac{1}{\theta_j} \sum_{m=0}^i \sum_{n=0}^j \frac{p_m^{(i)} p_n^{(j)}}{m+1} B(m+n+\beta+2, \alpha+1), \quad i, j = 0, 1, 2, \dots, N.$$

*Proof* Using Eq. (10) and orthogonality property of Jacobi polynomials one has:

$$P = \left( \int_0^x \Phi(t) dt, \Phi^T(x) \right)_{W^{(\alpha,\beta)}} \Delta^{-1},$$

where  $(\int_0^x \Phi(t) dt, \Phi^T(x))_{W^{(\alpha,\beta)}}$  and  $\Delta^{-1}$  are two  $(N+1) \times (N+1)$  matrices defined

as follows:

$$\begin{aligned} \left( \int_0^x \Phi(t) dt, \Phi^T(x) \right)_{W^{(\alpha, \beta)}} &= \left\{ \left( \int_0^x R_i^{(\alpha, \beta)}(t) dt, R_j^{(\alpha, \beta)}(x) \right)_{W^{(\alpha, \beta)}} \right\}_{i, j=0}^N, \\ &= \text{diag} \left\{ \frac{1}{\theta_j} \right\}_{j=0}^N. \end{aligned}$$

Now, set:

$$\begin{aligned} \rho_{ij} &= \left( \int_0^x R_i^{(\alpha, \beta)}(t) dt, R_j^{(\alpha, \beta)}(x) \right)_{W^{(\alpha, \beta)}} \\ &= \int_0^1 \left\{ \int_0^x R_i^{(\alpha, \beta)}(t) dt \right\} R_j^{(\alpha, \beta)}(x) W^{(\alpha, \beta)}(x) dx. \end{aligned}$$

$\int_0^x R_i^{(\alpha, \beta)}(t) dt$  and  $R_j^{(\alpha, \beta)}(x)$  by using Lemma 2.1 can be obtained as follows:

$$\begin{aligned} \int_0^x R_i^{(\alpha, \beta)}(t) dt &= \sum_{m=0}^i p_m^{(i)} \frac{x^{m+1}}{m+1}, \\ R_j^{(\alpha, \beta)}(x) &= \sum_{m=0}^j p_m^{(j)} x^m, \quad i, j = 0, 1, \dots, N. \end{aligned}$$

Therefore,  $\rho_{ij}$  by using Lemma 2.2 can be obtained as:

$$\begin{aligned} \rho_{ij} &= \sum_{m=0}^i \sum_{n=0}^j \frac{p_m^{(i)} p_n^{(j)}}{m+1} \int_0^1 x^{m+1} x^n (1-x)^\alpha x^\beta dx \\ &= \sum_{m=0}^i \sum_{n=0}^j \frac{p_m^{(i)} p_n^{(j)}}{m+1} B(m+n+\beta+2, \alpha+1). \end{aligned}$$

So, the elements of matrix  $P$  is obtained as:

$$P_{ij} = \frac{1}{\theta_j} \sum_{m=0}^i \sum_{n=0}^j \frac{p_m^{(i)} p_n^{(j)}}{m+1} B(m+n+\beta+2, \alpha+1), \quad i, j = 0, 1, 2, \dots, N.$$

■

Now the following theorem can present an upper bound for estimating the error of integral operator. First, the error vector  $E$  is defined as:

$$E = \int_0^x \Phi(t) dt - P\Phi(x) = [E_0, E_1, \dots, E_N],$$

where

$$E_k = \int_0^x R_k^{(\alpha, \beta)}(t) dt - \sum_{j=0}^N P_{kj} R_j^{(\alpha, \beta)}(x), \quad k = 0, 1, \dots, N.$$

**THEOREM 4.2** *If  $E_k = \int_0^x R_k^{(\alpha,\beta)}(t) dt - \sum_{j=0}^N P_{kj} R_j^{(\alpha,\beta)}(x) \in H_{W^{(\alpha,\beta)}}^r(\Omega)$ , then an error bound of integral operator of vector  $\Phi$  can be expressed by:*

$$\| E_k \|_{\mu, W^{(\alpha,\beta)}} \leq c^2(N(N+\alpha+\beta))^{\mu-r} \sum_{i=0}^k \sum_{j=0}^k \rho_i^{(k)} \rho_j^{(k)} B(i+j+\beta-r+3, \alpha+r+1).$$

*Proof* By using inequality (8), Lemma 2.1, and setting  $u(x) = \int_0^x R_k^{(\alpha,\beta)}(t) dt$  one has:

$$\begin{aligned} |u|_{r, W^{(\alpha,\beta)}} &= \left\| D^r \int_0^x R_k^{(\alpha,\beta)}(t) dt \right\|_{W^{(\alpha,\beta)}}^2 \\ &= \left\| D^r \left\{ \sum_{i=0}^k \frac{1}{i+1} p_i^{(k)} x^{i+1} \right\} \right\|_{W^{(\alpha+r,\beta+r)}}^2 \\ &= \left\| \sum_{i=0}^k \frac{i!}{\Gamma(i-r+2)} p_i^{(k)} x^{i-r+1} \right\|_{W^{(\alpha+r,\beta+r)}}^2 \\ &= \int_0^1 W^{(\alpha+r,\beta+r)}(x) \left( \sum_{i=0}^k \rho_i^{(k)} x^{i-r+1} \right) \left( \sum_{j=0}^k \rho_j^{(k)} x^{j-r+1} \right) dx \\ &= \sum_{i=0}^k \sum_{j=0}^k \rho_i^{(k)} \rho_j^{(k)} \int_0^1 (1-x)^{\alpha+r} x^{i+j+\beta-r+2} dx \\ &= \sum_{i=0}^k \sum_{j=0}^k \rho_i^{(k)} \rho_j^{(k)} B(i+j+\beta-r+3, \alpha+r+1), \end{aligned}$$

where  $\rho_i^{(k)} = \frac{i!}{\Gamma(i-r+2)} p_i^{(k)}$  and the theorem can be proved. ■

#### 4.2 The Jacobi Operational Matrix of Product

The following property of the product of two Jacobi function vector will also be applied to solve the Volterra and Volterra–Fredholm integro–differential equations.

$$\Phi(x) \Phi^T(x) Y \simeq \tilde{Y} \Phi(x), \tag{11}$$

where  $\tilde{Y}$  is a  $(N + 1) \times (N + 1)$  product operational matrix and its entries are determined in terms of the vector  $Y$ 's components. By using Eq.(11) and by the orthogonality property of the Jacobi polynomials the entries  $\tilde{Y}_{ij}$  can be calculated

as follows:

$$\begin{aligned}\tilde{Y}_{ij} &= \frac{1}{\theta_j} \sum_{k=0}^N Y_k \int_0^1 (\Phi(x))_i (\Phi(x))_j (\Phi(x))_k W^{(\alpha,\beta)}(x) dx \\ &= \frac{1}{\theta_j} \sum_{k=0}^N Y_k \int_0^1 R_i^{(\alpha,\beta)}(x) R_j^{(\alpha,\beta)}(x) R_k^{(\alpha,\beta)}(x) W^{(\alpha,\beta)}(x) dx \\ &= \frac{1}{\theta_j} \sum_{k=0}^N Y_k h_{ijk},\end{aligned}$$

where

$$h_{ijk} = \int_0^1 R_i^{(\alpha,\beta)}(x) R_j^{(\alpha,\beta)}(x) R_k^{(\alpha,\beta)}(x) W^{(\alpha,\beta)}(x) dx.$$

## 5. Applications of Operational Matrices of Integration and Product

In this section, the presented operational matrices are applied to solve the system of linear and nonlinear Fredholm, Volterra and Volterra–Fredholm integro–differential equations.

### 5.1 The System of Fredholm–Volterra Integor–Differential Equations

In this paper, a system of Fredholm–Volterra integro–differential equations is considered as follows:

$$\begin{aligned}u_i^{(m)}(x) &+ \sum_{k=1}^{m_1} F_{ik}(x, u_1(x), u_1'(x), \dots, u_1^{(m)}(x), \dots, u_n(x), \dots, u_n^{(m)}(x)) \\ &+ \sum_{j=1}^{m_2} \int_0^x k_{ij}(x, t) G_{ij}(u_1(t), u_1'(t), \dots, u_1^{(m)}(t), \dots, u_n(t), \dots, u_n^{(m)}(t)) dt \\ &+ \sum_{l=1}^{m_3} \int_0^1 h_{il}(x, t) L_{il}(u_1(t), u_1'(t), \dots, u_1^{(m)}(t), \dots, u_n(t), \dots, u_n^{(m)}(t)) dt \\ &= f_i(x), \quad 0 \leq x \leq 1, \quad i = 1, 2, \dots, n.\end{aligned}\tag{12}$$

Where  $k_{ij}(x, t)$  and  $h_{il}(x, t) \in L^2([0, 1] \times [0, 1])$ ,  $f_i$  are known functions, and  $G_{ij}$  and  $L_{il}$  are linear or nonlinear functions in terms of unknown functions  $u_1(x), u_2(x), \dots, u_n(x)$  and their derivatives. Consider system (12) with the following conditions:

$$u_i^{(s)}(x) = a_{ij}, \quad i = 1, \dots, n, \quad s = 0, 1, \dots, m - 1.$$

To solve system (12), the functions  $u_i(x), u_i^{(r)}(x), G_{ij}(t), H_{il}(t), k_{ij}(x, t)$ , and  $h_{il}(x, t)$  can be approximated as follows:

First it is assumed the unknown functions  $u_i^{(m)}(x)$ ,  $i = 1, \dots, n$ , are approximated

in the following forms:

$$u_i^{(m)}(x) \simeq \Phi^T(x) C_i. \tag{13}$$

The iterative integrating leads to:

$$u_i^{(s)}(x) \simeq \Phi^T(x) (P^{m-s})^T C_i + \sum_{j=0}^{m-1-s} a_{is} \frac{x^j}{j!}, \quad i = 1, \dots, n, \quad s = 0, \dots, m - 1. \tag{14}$$

Using Eqs.(13) and (14) other terms will be considered as the following general expansions:

$$\begin{aligned} F_{ik}(x, u_1(x), u_1'(x), \dots, u_1^{(m)}(x), \dots, u_n(x), \dots, u_n^{(m)}(x)) &\simeq \Phi^T(x) X_{ik}, \\ G_{ij}(u_1(t), u_1'(t), \dots, u_1^{(m)}(t), \dots, u_n(t), \dots, u_n^{(m)}(t)) &\simeq \Phi^T(x) Y_{ij}, \\ L_{il}(u_1(t), u_1'(t), \dots, u_1^{(m)}(t), \dots, u_n(t), \dots, u_n^{(m)}(t)) &\simeq \Phi^T(x) Z_{ij}, \\ k_{ij}(x, t) \simeq \Phi^T(x) K_{ij} \Phi(t), \quad h_{il}(x, t) &\simeq \Phi^T(x) H_{il} \Phi(t), \\ i = 1, \dots, n, \quad k = 1, \dots, m_1, \quad j = 1, \dots, m_2, \quad l = 1, \dots, m_3, \end{aligned}$$

where  $X_{ik}, Y_{ij}$ , and  $Z_{il}$  are the column vectors of the components of unknown vectors  $C_i, i = 1, \dots, n$ , and  $K_{ij}$  and  $H_{il}$  are known matrices. Also,

$$C_i = [c_0^i, c_1^i, \dots, c_N^i]^T, \quad \Phi(x) = [R_0^{(\alpha,\beta)}(x), R_1^{(\alpha,\beta)}(x), \dots, R_N^{(\alpha,\beta)}(x)]^T.$$

Substituting above approximations into system (12), leads to the following algebraic system:

$$\begin{aligned} \Phi^T(x) C_i + \Phi^T(x) \sum_{k=1}^{m_1} X_{ik} + \Phi^T(x) \left\{ \sum_{j=1}^{m_2} K_{ij} \tilde{Y}_{ij} \right\} P \Phi(x) \\ + \Phi^T(x) D \sum_{l=1}^{m_3} H_{il} Z_{il} \approx f_i(x), \quad i = 1, 2, \dots, n, \end{aligned} \tag{15}$$

where  $\tilde{Y}_{ij}$  are operational matrices of product and their elements are in terms of the elements of vectors  $Y_{ij}$ ,  $P$  is operational matrix of integration, and  $D_{(N+1) \times (N+1)}$  is the following known matrix,

$$D = \int_0^1 \Phi(t) \Phi^T(t) dt.$$

The system (15) has  $n(N + 1)$  unknown coefficients  $c_j^i$ . So,  $n(N + 1)$  collocating points are needed to collocate. For this purpose, the first  $n(N + 1)$  roots of Jacobi polynomials  $R_{n(N+1)+1}^{(\alpha,\beta)}$  are applied and the equations are collocated at them. Unknown coefficients are determined with solving the resultant system of linear or nonlinear algebraic equations. Finally, the approximate solutions are obtained as

follows:

$$u_i(x) \simeq \Phi^T(x) (P^m)^T C_i + \sum_{j=0}^{m-1} a_{i0} \frac{x^j}{j!}, \quad i = 1, \dots, n.$$

### 5.2 System of Volterra Integro–Differential Equations

A system of Volterra integro–differential equations can be presented as follows:

$$\begin{aligned} u_i^{(m)}(x) + \sum_{k=1}^{m_1} F_{ik}(x, u_1(x), u_1'(x), \dots, u_1^{(m)}(x), \dots, u_n(x), \dots, u_n^{(m)}(x)) \\ + \sum_{j=1}^{m_2} \int_0^x k_{ij}(x, t) G_{ij}(u_1(t), u_1'(t), \dots, u_1^{(m)}(t), \dots, u_n(t), \dots, u_n^{(m)}(t)) dt = f_i(x), \\ 0 \leq x \leq 1, \quad i = 1, 2, \dots, n. \end{aligned} \quad (16)$$

By using the approximate relations in subsection 5.1, one has:

$$\Phi^T(x) C_i + \Phi^T(x) \sum_{k=1}^{m_1} X_{ik} + \Phi^T(x) \left\{ \sum_{j=1}^{m_2} K_{ij} \tilde{Y}_{ij} \right\} P \Phi(x) \approx f_i(x). \quad (17)$$

By using the first  $n(N+1)$  roots of Jacobi polynomials  $R_{n(N+1)+1}^{(\alpha, \beta)}(x)$  and collocated system (17), unknown coefficients  $c_j^i$  are determined.

### 5.3 System of Fredholm Integro–Differential Equations

A system of Fredholm integro–differential equations can be presented as follows,

$$\begin{aligned} u_i^{(m)}(x) + \sum_{k=1}^{m_1} F_{ik}(x, u_1(x), u_1'(x), \dots, u_1^{(m)}(x), \dots, u_n(x), \dots, u_n^{(m)}(x)) \\ + \sum_{j=1}^{m_2} \int_0^1 k_{ij}(x, t) G_{ij}(u_1(t), u_1'(t), \dots, u_1^{(m)}(t), \dots, u_n(t), \dots, u_n^{(m)}(t)) dt \\ = f_i(x), \quad 0 \leq x \leq 1, \quad i = 1, 2, \dots, n. \end{aligned} \quad (18)$$

By using the approximate relations in subsection 5.1, one has:

$$\Phi^T(x) C_i + \Phi^T(x) \sum_{k=1}^{m_1} X_{ik} + \Phi^T(x) \sum_{j=1}^{m_2} K_{ij} D Y_{ij} \approx f_i(x). \quad (19)$$

By using the first  $n(N+1)$  roots of Jacobi polynomials  $R_{n(N+1)+1}^{(\alpha, \beta)}(x)$  and collocated system (19), unknown coefficients  $c_j^i$  are determined.

## 6. Illustrative Examples

In this section, some systems of integro–differential equations are considered and solved by the proposed method. Comparison between the results of present method

with the corresponding analytic solutions are given. For this purpose, the maximum of absolute error is computed.

*Example 6.1* Consider the following nonlinear Fredholm–Volterra integro-differential equation.

$$x^4 u^{(6)}(x) + u^{(3)}(x) + u'(x) = f(x) - 2 \int_0^x (1 + u^2(t))dt + \int_0^1 e^t u^3(t)dt, \quad (20)$$

where

$$f(x) = -x^4 \cos(x) + 0.5 \sin(2x) + 3x + 0.4 - 0.1e \left( (\cos(1) + \sin(1))(\cos^2(1) + 3e) \right).$$

Subject to the following conditions,

$$u(0) = u^{(4)}(0) = 1, \quad u'(0) = u'''(0) = u^{(5)}(0) = 0, \quad u''(0) = -1.$$

The exact solution is  $u(x) = \cos(x)$ . By the applying the technique described in pervious section with  $N = 7$ , unknown functions and kernels are approximated as:

$$\begin{aligned} u^{(6)}(x) &\simeq \Phi^T(x)C, & u^{(5)}(x) &\simeq \Phi^T(x)P^T C, \\ u^{(4)}(x) &\simeq \Phi^T(x)(P^2)^T C + 1 \simeq \Phi^T(x)(P^2)^T C + \Phi^T(x)U_1, \\ u'''(x) &\simeq \Phi^T(x)(P^3)^T C + x \simeq \Phi^T(x)(P^3)^T C + \Phi^T(x)U_2, \\ u''(x) &\simeq \Phi^T(x)(P^4)^T C + \frac{x^2}{2} - 1 \simeq \Phi^T(x)(P^4)^T C + \Phi^T(x)U_3, \\ u'(x) &\simeq \Phi^T(x)(P^5)^T C + \frac{x^3}{6} - x \simeq \Phi^T(x)(P^5)^T C + \Phi^T(x)U_4, \\ u(x) &\simeq \Phi^T(x)(P^6)^T C + \frac{x^4}{24} - \frac{x^2}{2} + 1 \simeq \Phi^T(x)(P^6)^T C + \Phi^T(x)U_5, \\ 1 &\simeq \Phi^T(x) K_1 \Phi(t), & e^t &\simeq \Phi^T(x) K_2 \Phi(t), \\ 1 + u^2(t) &\simeq \Phi^T(x)X, & u^3(t) &\simeq \Phi^T(x)Y. \end{aligned}$$

By using above approximations, the equation (20) is rewritten as:

$$\begin{aligned} x^4 \Phi^T(x) C + \Phi^T(x)(P^3)^T C + \Phi^T(x)U_2 + \Phi^T(x)(P^5)^T C + \Phi^T(x)U_4 \\ + 2\Phi^T(x)K_1 \tilde{X} P \Phi(x) - \Phi^T(x)K_2 DY \approx f(x). \end{aligned} \quad (21)$$

Now, using the roots of  $R_s^{(\alpha, \beta)}(x)$  and collocating the system (21), reduces the problem to solve a system of algebraic equations. Unknown coefficients and thereupon the approximate solutions are obtained for some values of parameters  $\alpha$  and  $\beta$  as follows:

$\alpha = 0, \beta = 0 :$

$$\begin{aligned} c_0 &= -0.84146897152585042168, & c_1 &= 0.23377113925620035053, \\ c_2 &= 0.071827849955133033300, & c_3 &= -0.0039213111323349086308, \\ c_4 &= -0.00054119755921478355657, & c_5 &= 0.000038508187363987792142 \end{aligned}$$

$$c_6 = -0.000014381333191174830100, \quad c_7 = 0.0000074576369234027344279,$$

$$\begin{aligned} u(x) &\simeq \Phi^T(x)(P^6)^T C + \frac{x^4}{24} - \frac{x^2}{2} + 1 \\ &= 0.9999999832254307294 - 0.50000214580659308841x^2 \\ &\quad + 0.000015864714283463276531x^3 + 0.00012642429426326671848x^5 \\ &\quad - 0.0015388985051473808897x^6 + 0.000094330653195026923364x^7 \\ &\quad + 1.2169520412731968222 \times 10^{-7}x + 0.041606606150062886033x^4. \end{aligned}$$

$\alpha = -\frac{1}{2}, \beta = -\frac{1}{2} :$

$$\begin{aligned} c_0 &= -0.82359036680648545713, & c_1 &= 0.46462461440034186335, \\ c_2 &= 0.14318948965683630951, & c_3 &= -0.0077925534859049611932, \\ c_4 &= -0.0011116533857470159668, & c_5 &= 0.000098402973547378049552, \\ c_6 &= -0.41170291119818802229e - 4, & c_7 &= 0.000020282938324986794400, \\ u(x) &\simeq 8.464635915202863 \times 10^{-8}x + 0.9999999934151843735 \\ &\quad + 0.000014468749100100495715x^3 + 0.041609254326484002152x^4 \\ &\quad + 0.00012399501368439068203x^5 - 0.0015379789911230361779x^6 \\ &\quad + 0.000094273820796129258949x^7 - 0.50000179274958999233x^2. \end{aligned}$$

$\alpha = \frac{1}{2}, \beta = \frac{1}{2} :$

$$\begin{aligned} c_0 &= -0.85043901614166469209, & c_1 &= 0.15649949733471769683, \\ c_2 &= 0.043199582072983930866, & c_3 &= -0.0022495205396915608514, \\ c_4 &= -0.00029757421490597881067, & c_5 &= 0.000020302675934040396994, \\ c_6 &= -0.0000087308839336630478452, & c_7 &= 0.0000050893934639569033884, \\ u(x) &\simeq 0.99999999703368038824 - 0.50000247819439319704x^2 \\ &\quad + 0.000094380386337730722662x^7 + 1.5982903158795169636 \times 10^{-7}x \\ &\quad + 0.000017124360642998974771x^3 - 0.15397041793127890336e - 2x^6 \\ &\quad + 0.12856048907421420252e - 3x^5 + 0.041604261752377567277x^4. \end{aligned}$$



$$\alpha = \frac{1}{2}, \quad \beta = -\frac{1}{2} :$$

$$\begin{aligned} c_0 &= -0.93974743489940852880, & c_1 &= 0.17860847577677168587, \\ c_2 &= 0.074839629911566906680, & c_3 &= -0.0033624458261905494574, \\ c_4 &= -0.00070950214849274811835, & c_5 &= 0.00022108065083583251016 \\ c_6 &= -0.00018170906069341951844, & c_7 &= 0.00010948150899525003352, \\ u(x) &\simeq 0.9999999966509439337 - 0.50000113769580626249x^2 \\ &+ 4.8400776075473091969 \times 10^{-8}x + 0.000010090793828792263055x^3 \\ &+ 0.00010196073522603268356x^5 - 0.0015205554983298800759x^6 \\ &+ 0.000088899135119744677522x^7 + 0.041623026474336101908x^4. \end{aligned}$$

$$\alpha = -\frac{1}{2}, \quad \beta = \frac{1}{2} :$$

$$\begin{aligned} c_0 &= -0.70743336058984757875, & c_1 &= 0.28601258188112225667, \\ c_2 &= 0.06833865669819777907, & c_3 &= -0.0043758488231797072075, \\ c_4 &= -0.00050997253648574598916, & c_5 &= 0.000020280942452854649391, \\ c_6 &= 0.0000028017083221428242500, & c_7 &= -0.0000028018918773302405027, \\ u(x) &\simeq 0.99999999449877493485 - 0.50000377146074509279x^2 \\ &+ 0.041586124550645938692x^4 + 2.6728592702944422416 \times 10^{-7}x \\ &+ 0.000023914332830985303442x^3 + 0.00015432456103902070638x^5 \\ &- 0.0015582491666549713612x^6 + 0.000099687294013215775697x^7. \end{aligned}$$

$$\alpha = \frac{1}{10}, \quad \beta = \frac{1}{10} :$$

$$\begin{aligned} c_0 &= -0.84370920460982268323, & c_1 &= 0.21272426884664085882, \\ c_2 &= 0.064165988230373531298, & c_3 &= -0.0034781977847499708353, \\ c_4 &= -0.00047622185581651942936, & c_5 &= 0.000033082669919090350699, \\ c_6 &= -0.000012399605048553187536, & c_7 &= 0.0000065616780076748106867, \\ u(x) &\simeq 1.2927471941717776429 \times 10^{-7}x + 0.000016126921556826295087x^3 \\ &- 0.50000221396643333788x^2 + 0.041606114856013658415x^4 \\ &+ 0.00012687311458595242886x^5 - 0.0015390681142481039356x^6 \\ &+ 0.000094341146079419796908x^7 + 0.99999999808358551636. \end{aligned}$$

Table 1. Maximum absolute error for  $N = 7$  and different values of  $\alpha$  and  $\beta$  for Example 6.1.

$\alpha$	$\beta$	$Error(u(x))$	$\alpha$	$\beta$	$Error(u(x))$
0	0	$3.3610 \times 10^{-9}$	$-\frac{1}{2}$	$\frac{1}{2}$	$1.3010 \times 10^{-8}$
$-\frac{1}{2}$	$-\frac{1}{2}$	$1.4901 \times 10^{-9}$	$\frac{1}{10}$	$\frac{1}{10}$	$3.4977 \times 10^{-9}$
$\frac{1}{2}$	$\frac{1}{2}$	$3.1375 \times 10^{-9}$	1	1	$2.7844 \times 10^{-9}$
$\frac{1}{2}$	$-\frac{1}{2}$	$2.6802 \times 10^{-8}$	$-\frac{3}{4}$	$-\frac{3}{4}$	$9.1324 \times 10^{-10}$

$\alpha = 1, \beta = 1 :$

$$\begin{aligned}
c_0 &= -0.85583393504609201494, & c_1 &= 0.11772277077437384702, \\
c_2 &= 0.028855108852909964458, & c_3 &= -0.0014111922191631499739, \\
c_4 &= -0.00017805265479934357050, & c_5 &= 0.000013944777008855055759, \\
c_6 &= -0.0000082223759645466613808, & c_7 &= 0.0000052218116967253402439, \\
u(x) &\simeq 0.99999999555495728346 - 0.50000278969402280865x^2 \\
&+ 0.041602166809956211835x^4 + 1.9802433185087373394 \times 10^{-7}x \\
&+ 0.000018266527228444148422x^3 + 0.00013045599090272277424x^5 \\
&- 0.0015404132204288210006x^6 + 0.000094423509321685317870x^7.
\end{aligned}$$

$\alpha = -\frac{3}{4}, \beta = -\frac{3}{4} :$

$$\begin{aligned}
c_0 &= -0.80576825479856602497, & c_1 &= 0.92512334018663338525, \\
c_2 &= 0.22860574347226291513, & c_3 &= -0.011811038590888473173, \\
c_4 &= -0.0016975549984185060258, & c_5 &= 0.00017380438500993731499, \\
c_6 &= -0.000078140949943786055151e, & c_7 &= 0.000038023047897313204674, \\
u(x) &\simeq 6.6995650794672600193 \times 10^{-8}x + 0.9999999972148770080 \\
&- 0.50000160904417864069x^2 + 0.041610712757527487058x^4 \\
&+ 0.000013713254854022297860x^3 + 0.0001226496975817763947x^5 \\
&- 0.0015374693822965339567x^6 + 0.000094242465017402172309x^7.
\end{aligned}$$

Maximum absolute error for  $N = 7$  and different values of  $\alpha$  and  $\beta$  are listed in Table 1.

*Example 6.2* Consider the following system of linear Fredholm integro-differential equations.

$$\begin{cases} u''(x) + v'(x) + \int_0^1 2xt(u(t) - 3v(t)) dt = 3x^2 + \frac{3}{10}x + 8, \\ v''(x) + u'(x) + \int_0^1 3(2x + t^2)(u(t) - 2v(t)) dt = 21x + \frac{4}{5}, \end{cases} \quad (22)$$

subject to initial conditions  $u(0) = 1, u'(0) = 0, v(0) = -1, v'(0) = 2$ , and the exact solutions are  $u(x) = 3x^2 + 1$  and  $v(x) = x^3 + 2x - 1$ . With  $N = 5$ , the solutions

Table 2. Maximum absolute error for  $N = 5$  and different values of  $\alpha$  and  $\beta$  for Example 6.2.

$\alpha$	$\beta$	$Error(u(x))$	$Error(v(x))$
0	0	$9.4140 \times 10^{-18}$	$1.1259 \times 10^{-15}$
$-\frac{1}{2}$	$-\frac{1}{2}$	$2.2874 \times 10^{-17}$	$1.6519 \times 10^{-15}$
$\frac{1}{2}$	$\frac{1}{2}$	$5.2144 \times 10^{-17}$	$1.5230 \times 10^{-15}$
$\frac{1}{2}$	$-\frac{1}{2}$	$1.9466 \times 10^{-16}$	$5.0418 \times 10^{-16}$
$-\frac{1}{2}$	$\frac{1}{2}$	$8.4567 \times 10^{-16}$	$1.4648 \times 10^{-14}$
$\frac{1}{10}$	$\frac{1}{10}$	$1.1842 \times 10^{-16}$	$1.8589 \times 10^{-16}$
1	1	$1.0272 \times 10^{-16}$	$1.0774 \times 10^{-15}$
$-\frac{3}{4}$	$-\frac{3}{4}$	$4.3710 \times 10^{-17}$	$3.0863 \times 10^{-16}$

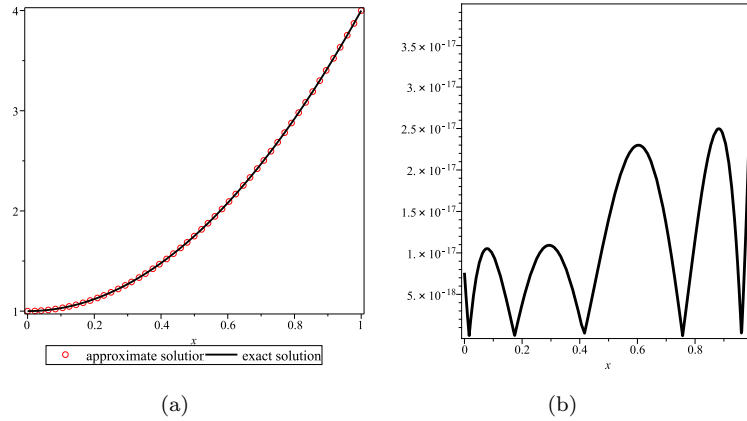


Figure 1. (a) Comparison of exact and approximate solutions, (b) absolute error function for  $u(x)$ ,  $\alpha = -\frac{1}{2}$ ,  $\beta = \frac{1}{2}$ , and  $N = 10$  for Example 6.2.

and kernels are approximated as:

$$\begin{aligned}
 u''(x) &\simeq \Phi^T(x)C_1, & u'(x) &\simeq \Phi^T(x)P^T C_1, \\
 u(x) &\simeq \Phi^T(x)(P^2)TC_1 + 1 \simeq \Phi^T(x)(P^2)^T C_1 + \Phi^T(x)U_1, \\
 v''(x) &\simeq \Phi^T(x)C_2, & v'(x) &\simeq \Phi^T(x)P^T C_2 + 2 \simeq \Phi^T(x)P^T C_2 + \Phi^T(x)U_2, \\
 v(x) &\simeq \Phi^T(x)(P^2)^T C_2 + 2x - 1 \simeq \Phi^T(x)(P^2)^T C_2 + \Phi^T(x)U_3, \\
 2xt &\simeq \Phi^T(x)K_1\Phi(t), & 3(2x + t^2) &\simeq \Phi^T(x)K_2\Phi(t).
 \end{aligned}$$

By using above relations the system (22) is rewritten as follows:

$$\begin{cases}
 \Phi^T(x)\{C_1 + P^T C_2 + U_2 + K_1 D(P^2)^T C_1 + K_1 D U_1 - 3K_1 D((P^2)^T C_2 + U_3)\} \\
 \approx 3x^2 + \frac{3}{10}x + 8, \\
 \Phi^T(x)\{C_2 + P^T C_1 + K_2 D((P^2)^T C_1 + U_1 - 2(P^2)^T C_2 + U_3)\} \approx 21x + \frac{4}{5},
 \end{cases} \tag{23}$$

Now, using the roots of  $R_{13}^{(\alpha,\beta)}(x)$  and collocating the system (23), reduces the problem to solve a system of linear algebraic equations and unknown coefficients are obtained for some values of parameters  $\alpha$  and  $\beta$ . Table 2 displays the maximum absolute errors for various values  $\alpha$  and  $\beta$  parameters with  $N = 5$ . Comparison of the exact and approximate solutions and the plot of absolute error functions are presented in Figures 1 and 2 for  $\alpha = \beta = -\frac{1}{2}$ . The figures show the good agreement between the exact and approximate solutions.

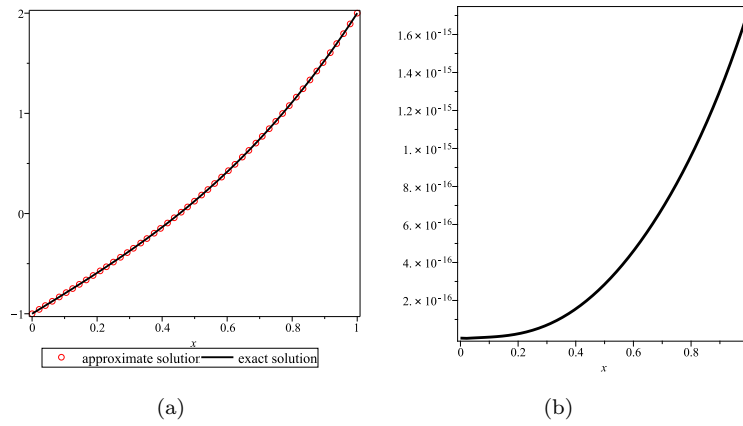


Figure 2. (a) Comparison of exact and approximate solutions, (b) absolute error function for  $v(x)$ ,  $\alpha = -\frac{1}{2}$ ,  $\beta = \frac{1}{2}$ , and  $N = 10$  for Example 6.2.

*Example 6.3* Third example covers the system of nonlinear Volterra integro-differential equation.

$$\begin{cases} u'''(x) = x - u'(x) - \int_0^x (u''^2(t) + v''^2(t)) dt, \\ v'''(x) = \sin(x) + \frac{1}{2} \sin^2(x) + \int_0^x u''(t) v(t) dt, \end{cases} \quad (24)$$

subject to initial conditions  $u(0) = 0, u'(0) = 1, u''(0) = 0, v(0) = 0, v'(0) = 0, v''(0) = -1$  and the exact solutions are  $u(x) = \sin(x)$  and  $v(x) = \cos(x)$ . With  $N = 7$ , the unknown functions and kernels are approximated as:

$$\begin{aligned} u'''(x) &\simeq \Phi^T(x)C_1, & u''(x) &\simeq \Phi^T(x)P^T C_1, \\ u'(x) &\simeq \Phi^T(x)(P^2)TC_1 + 1 \simeq \Phi^T(x)(P^2)^T C_1 + \Phi^T(x)U_1, \\ u(x) &\simeq \Phi^T(x)(P^3)TC_1 + x \simeq \Phi^T(x)(P^3)^T C_1 + \Phi^T(x)U_2, \\ v'''(x) &\simeq \Phi^T(x)C_2, & v''(x) &\simeq \Phi^T(x)P^T C_2 - 1 \simeq \Phi^T(x)P^T C_2 + \Phi^T(x)U_3, \\ v'(x) &\simeq \Phi^T(x)(P^2)TC_2 - x \simeq \Phi^T(x)(P^2)^T C_2 + \Phi^T(x)U_4, \\ v(x) &\simeq \Phi^T(x)(P^3)TC_2 - \frac{x^2}{2} + 1 \simeq \Phi^T(x)(P^3)^T C_2 + \Phi^T(x)U_5, \\ 1 &\simeq \Phi^T(x)K\Phi(t), & u''^2(x) + v''^2(x) &\simeq \Phi^T(x)X_1, & u''(t) v(t) &\simeq \Phi^T(x)X_2 \end{aligned}$$

Using above approximations leads to the following nonlinear systems:

$$\begin{cases} \Phi^T(x)\{C_1 + (P^2)^T C_1 + U_1 + K\tilde{X}_1 P \Phi(x)\} \approx x, \\ \Phi^T(x)\{C_2 - K\tilde{X}_2 P \Phi(x)\} \approx \sin(x) + \frac{1}{2} \sin^2(x), \end{cases} \quad (25)$$

where  $P$  is Jacobi operational matrix of integration and  $\tilde{X}_1$  and  $\tilde{X}_2$  are operational matrices of product which their entries are obtained in terms of components of vectors  $X_1$  and  $X_2$ . Using the roots of  $R_{17}^{(\alpha, \beta)}(x)$  and collocating the system (25), the problem reduces to solve a system of nonlinear algebraic equations which will be solved by means of Newton iterative method and unknown coefficients are determined for some values of  $\alpha$  and  $\beta$  parameters. By solving algebraic system obtained, unknown coefficients are determined. Table 3 displays the maximum ab-

Table 3. Maximum absolute error for  $N = 7$  and different values of  $\alpha$  and  $\beta$  for Example 6.3.

$\alpha$	$\beta$	$Error(u(x))$	$Error(v(x))$
0	0	$1.2946 \times 10^{-7}$	$9.8211 \times 10^{-7}$
$-\frac{1}{2}$	$-\frac{1}{2}$	$1.2992 \times 10^{-7}$	$9.5397 \times 10^{-7}$
$\frac{1}{2}$	$\frac{1}{2}$	$1.2859 \times 10^{-7}$	$9.4792 \times 10^{-7}$
$\frac{1}{2}$	$-\frac{1}{2}$	$1.2044 \times 10^{-7}$	$8.8936 \times 10^{-7}$
$-\frac{1}{2}$	$\frac{1}{2}$	$1.3346 \times 10^{-7}$	$9.8152 \times 10^{-7}$
$\frac{1}{10}$	$\frac{1}{10}$	$1.2930 \times 10^{-7}$	$9.5126 \times 10^{-7}$
1	1	$1.2709 \times 10^{-7}$	$9.3987 \times 10^{-7}$
$-\frac{3}{4}$	$-\frac{3}{4}$	$1.2990 \times 10^{-7}$	$9.5346 \times 10^{-7}$

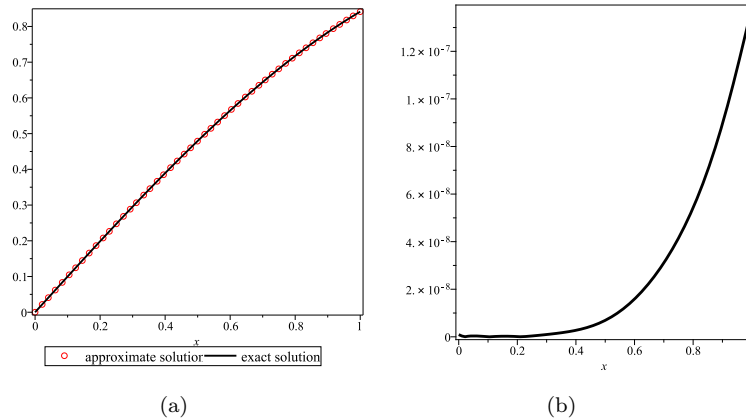


Figure 3. (a) Comparison of exact and approximate solutions, (b) absolute error function for  $u(x)$  of Example 6.3 for for  $\alpha = \beta = 0$  and  $N = 7$ .

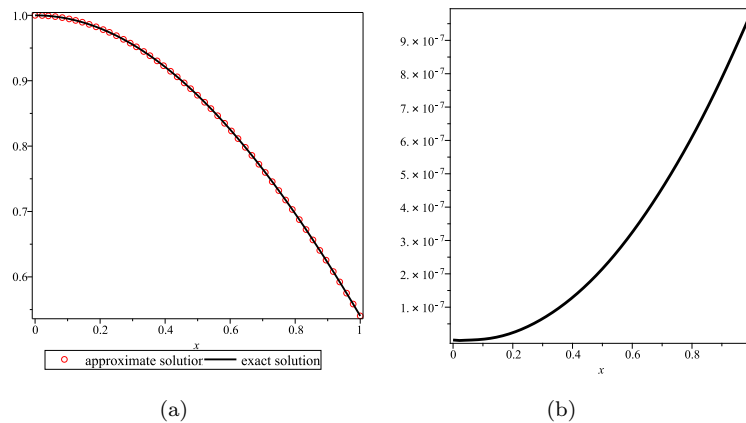


Figure 4. (a) Comparison of exact and approximate solutions, (b) absolute error function for  $v(x)$  of Example 6.3 for for  $\alpha = \beta = 0$  and  $N = 7$ .

solute errors for some values of  $\alpha$  and  $\beta$  parameters for  $N = 7$ . Also, the parts (a) of Figures 3 and 4 show the comparison between the exact and approximate solutions and the parts (b) of these figures display the absolute error functions for  $\alpha = \beta = 0$ .

*Example 6.4* In this example, a nonlinear Volterra integro-differential equation is considered.

Table 4. Maximum absolute error for  $N = 7$  and different values of  $\alpha$  and  $\beta$  for Example 6.4.

$\alpha$	$\beta$	$Error(u(x))$	$Error(v(x))$	$Error(w(x))$
0	0	$3.8745 \times 10^{-15}$	$1.7777 \times 10^{-14}$	$1.3009 \times 10^{-13}$
$-\frac{1}{2}$	$-\frac{1}{2}$	$2.2631 \times 10^{-13}$	$2.5380 \times 10^{-13}$	$2.5361 \times 10^{-12}$
$\frac{1}{2}$	$\frac{1}{2}$	$1.7667 \times 10^{-13}$	$2.3190 \times 10^{-14}$	$4.0038 \times 10^{-13}$
$-\frac{1}{2}$	$-\frac{1}{2}$	$1.7142 \times 10^{-13}$	$1.2179 \times 10^{-13}$	$1.4473 \times 10^{-12}$
$-\frac{1}{2}$	$\frac{1}{2}$	$1.0595 \times 10^{-12}$	$1.8054 \times 10^{-12}$	$1.6017 \times 10^{-11}$
$\frac{3}{4}$	$\frac{3}{4}$	$2.2631 \times 10^{-13}$	$2.5380 \times 10^{-13}$	$2.5362 \times 10^{-13}$
$\frac{4}{10}$	$\frac{4}{10}$	$3.2159 \times 10^{-13}$	$5.5133 \times 10^{-13}$	$4.9081 \times 10^{-12}$

$$\begin{cases} u''(x) = x + 2x^3 + 2v'^2(x) - \int_0^x (v'^2(t) + u(t)w''(t))dt, \\ v''(x) = -3x^2 - xu(x) + \int_0^x (xtv'(t)u''(t) + w'(t))dt, \\ w''(x) = 2 - \frac{4}{3}x^3 + u''^2(x) - 2u^2x + \int_0^x (x^2v(t) + u'^2(t) + t^3w''(t))dt, \end{cases} \quad (26)$$

subject to initial conditions  $u(0) = u'(0) = v(0) = w(0) = w'(0) = 0, v'(0) = 1$  and the exact solutions are  $u(x) = x^2, v(x) = x$  and  $w(x) = 3x^2, 0 \leq x \leq 1$ . With  $N = 7$ , the unknown functions and kernels are approximated as:

$$\begin{aligned} u''(x) &\simeq \Phi^T(x)C_1, & u'(x) &\simeq \Phi^T(x)P^T C_1, & u(x) &\simeq \Phi^T(x)(P^2)TC_1 \\ v''(x) &\simeq \Phi^T(x)C_2, & v'(x) &\simeq \Phi^T(x)P^T C_2 + 1 \simeq \Phi^T(x)P^T C_2 + \Phi^T(x)U_1, \\ v(x) &\simeq \Phi^T(x)(P^2)^T C_2 + x \simeq \Phi^T(x)P^2)^T C_2 + \Phi^T(x)U_2, \\ w''(x) &\simeq \Phi^T(x)C_3, & w'(x) &\simeq \Phi^T(x)P^T C_3, & w(x) &\simeq \Phi^T(x)(P^2)TC_3 \\ v'^2(x) &\simeq \Phi^T(x)X_1, & u(x)w''(x) &\simeq \Phi^T(x)X_2, & v'(x)u''(x) &\simeq \Phi^T(x)X_3 \\ u'^2(x) &\simeq \Phi^T(x)X_4, & u''^2(x) &\simeq \Phi^T(x)X_5, & u^2(x) &\simeq \Phi^T(x)X_6 \\ 1 &\simeq \Phi^T(x)K_1\Phi(t), & xt &\simeq \Phi^T(x)K_2\Phi(t), & x^2 &\simeq \Phi^T(x)K_3\Phi(t), \\ t^3 &\simeq \Phi^T(x)K_4\Phi(t). \end{aligned}$$

By using above relations, the system (26) is rewritten as:

$$\begin{cases} \Phi^T(x)\{C_1 + K_1\tilde{X}_1P\Phi(x) + K_1\tilde{X}_2P\Phi(x) - 2X_1\} = x + 2x^3, \\ \Phi^T(x)\{C_2 + x(P^2)^T C - 1 - K_2\tilde{X}_3P\Phi(x) - K_1\tilde{Y}_1P\Phi(x)\} = -3x^2, \\ \Phi^T(x)\{C_3 - X_5 + 2X_6 - K_3\tilde{Y}_2P\Phi(x) - K_3\tilde{U}_2P\Phi(x) - K_1\tilde{X}_4P\Phi(x) \\ - K_4\tilde{C}_3\Phi(x)\} = 2 - \frac{4}{3}x^3, \end{cases} \quad (27)$$

where  $\tilde{Y}_1$  and  $\tilde{Y}_2$  are operational matrices of product which their entries are obtained in terms of components of vectors  $Y_1 = P^T C_3$  and  $Y_2 = (P^2)^T C_2$ . Using the roots of  $R_{25}^{(\alpha, \beta)}(x)$  and collocating the system (27), the problem reduces to solve a system of nonlinear algebraic equations, which will be solved by means of Newton iterative method and unknown coefficients are determined for some values of  $\alpha$  and  $\beta$  parameters. Table 4 displays the maximum absolute errors for some values of parameters  $\alpha$  and  $\beta$  with  $N = 7$ . Also, the comparison between the exact and approximate solutions are displayed in the parts (a) of Figures 5–7 and the absolute error functions are indicated in parts of (b) of these figures for  $\alpha = \beta = 1$ .

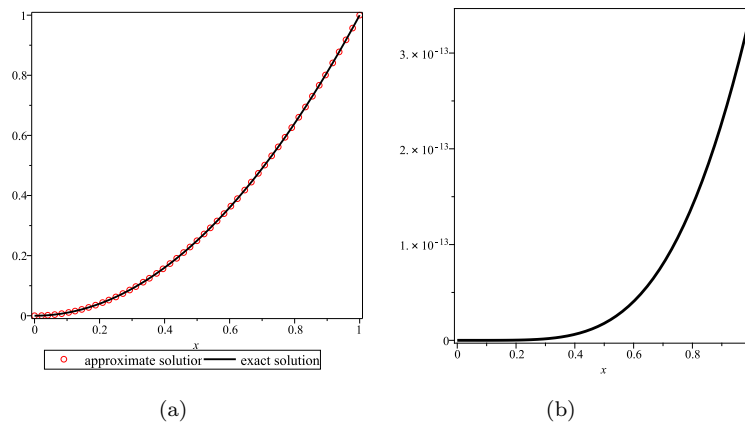


Figure 5. (a) Comparison of exact and approximate solutions, (b) absolute error function for  $u(x)$  of Example 6.4 for for  $\alpha = \beta = 1$  and  $N = 7$ .

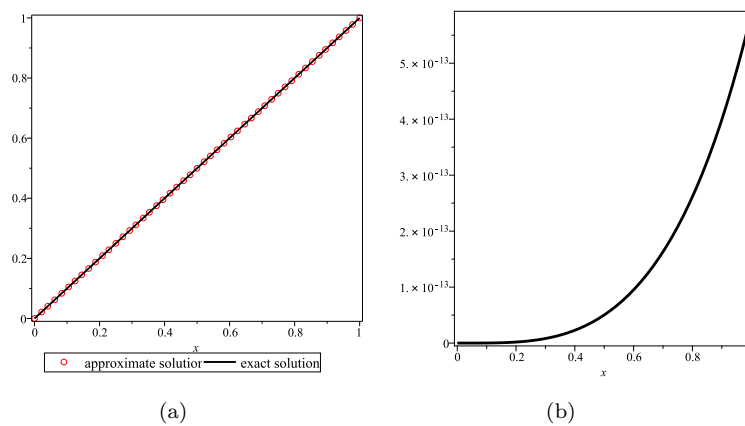


Figure 6. (a) Comparison of exact and approximate solutions, (b) absolute error function for  $v(x)$  of Example 6.4 for for  $\alpha = \beta = 1$  and  $N = 7$ .

## 7. Conclusion

In this paper, the shifted Jacobi collocation method was employed to solve a class of systems of Fredholm and Volterra integro–differential equations. First, a general formulation for the Jacobi operational matrix of integration has been derived. This matrix is used to approximate numerical solution of system of linear and nonlinear Volterra integro–differential equations. Proposed approach was based on the shifted Jacobi collocation method. The solutions obtained using the proposed method shows that this method is a powerful mathematical tool for solving the integro–differential equations. Proving the convergence of the method, consistency and stability are ensured automatically. Moreover, only a small number of shifted Jacobi polynomials is needed to obtain a satisfactory result.

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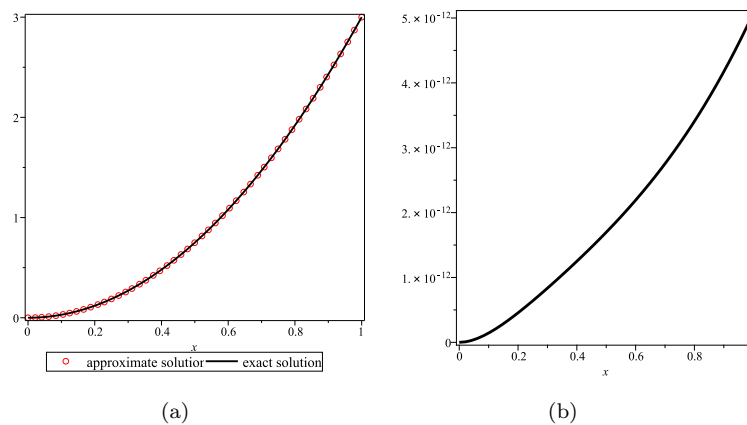


Figure 7. (a) Comparison of exact and approximate solutions, (b) absolute error function for  $w(x)$  of Example 6.4 for  $\alpha = \beta = 1$  and  $N = 7$ .

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