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Common Fixed-Point Theorems For Generalized Fuzzy Contraction Mapping

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Abstract In this paper we investigate common fixed point theorems for contraction mapping in fuzzy metric space introduced by Gregori and Sapena [V. Gregori, A. Sapena, On fixed-point theorems in fuzzy metric spaces, Fuzzy Sets and Systems, 125 (2002), 245-252].

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1. Introduction and Preliminaries

George and Veeramani [3] modified the concept of fuzzy metric space, introduced by Kramosil and Michalek and obtained several classical theorems on this new structure. Actually, this topology is first countable and metrizable [6]. Also the theory of fuzzy metric space is, in this context, very diferent from the classical theory of metric completion and metric best approximation, e.g. see [5, 6] and [1], respectively. Fixed point theory has important applications in diverse disciplines of mathematics, statistics, engineering and economics in dealing with problems arising in: approximation theory, potential theory, game theory, mathematical economics, etc. Several authors [4, 7–9, 11, 13] have proved fixed point theorems for contractions in fuzzy metric spaces, using one of the two different types of completeness: in the sense of Grabiec [4], or in the sense of Schweizer and Sklar [3, 12]. Gregori and Sapena [7, 13] introduced a new class of fuzzy contraction mappings and proved several fixed point theorems in fuzzy metric spaces. Gregori and Sapena's results

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extend classical Banach fixed point theorem and can be considered as a fuzzy version of Banach contraction theorem. In this paper, following the results of Gregori and Sapena we give a new common fixed point theorem in the two different types of completeness and by using the recent definition of contractive mapping of Gregori and Sapena [7] in fuzzy metric spaces.

Recall [12] that a continuous t-norm is a binary operation $* : [0,1]\times[0,1] \to [0,1]$ such that $([0,1], \leqslant, *)$ is an ordered Abelian topological monoid with unit 1. The two important t-norms, the minimum and the usual product, will be denoted by min and $\cdot$, respectively.

DEFINITION 1.1 *([3]) A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a non empty set, $*$ is a continuous t-norm and M is a fuzzy set of $X \times X \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X, s, t > 0$:*

(FM1) $M(x, y, t) > 0$;
(FM2) $M(x, y, t) = 1$ *if and only if* $x = y$;
(FM3) $M(x, y, t) = M(y, x, t)$;
(FM4) $M(x, z, t + s) \geqslant M(x, y, t) * M(y, z, s)$;
(FM5) $M(x, y, .) : (0, \infty) \to [0, 1]$ *is continuous.*

If, in the above definition, the triangular inequality(FM4) is replaced by

(NAF) $M(x, y, \max\{t, s\}) \geqslant M(x, z, t) * M(y, z, s) \qquad \forall x, y, z \in X, \forall t, s > 0,$

then the triple $(X, M, *)$ is called a non-Archimedean fuzzy metric space. It is easy to check that the triangular inequality (NAF) implies(FM4), that is, every non-Archimedean fuzzy metric space is itself a fuzzy metric space.

Example 1.2 (George and Veeramani[3]) Let (X, d) be a (non-Archimedean) metric space. Let M_d be the fuzzy set defined on $X \times X \times (0, +\infty)$ by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $(X, M_d, \min)$ is a (non-Archimedean) fuzzy metric space and called standard (non-Archimedean) fuzzy metric space.

Remark 1 ([3]) In fuzzy metric space $(X, M, *)$, $M(x, y, .)$ is non decreasing for all $x, y \in X$.

DEFINITION 1.3 *([4]) A sequence x_n in X is said to be convergent to a point x in X (denoted by $x_n \to x$), if $M(x_n, x, t) \to 1$,for all $t > 0$.*

DEFINITION 1.4 *Let $(X, M, *)$ be a fuzzy metric space.*

(a) A sequence $\{x_n\}$ is called G-Cauchy if $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ for each $t > 0$ and $p \in \mathbb{N}$. The fuzzy metric space $(X, M, *)$ is called G-complete if every G-Cauchy sequence is convergent [7].
(b) A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is a Cauchy sequence if for each $\epsilon \in (0, 1)$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1-\epsilon$, for all $n, m \geqslant n_0$. The fuzzy metric space $(X, M, *)$ is called complete if every Cauchy sequence is convergent [3].

PROPOSITION 1.5 *([7])*

(a) The sequence $\{x_n\}$ in the metric space X is contractive in (X, d) iff $\{x_n\}$ is fuzzy contractive in the induced fuzzy metric space $(X, M_d, *)$.
(b) The standard fuzzy metric space $(X, M_d, \min)$ is complete iff the metric space (X, d) is complete.

(c) If sequence $\{x_n\}$ is fuzzy contractive in $(X, M, *)$ then it is G-Cauchy.

Remark 2 ([10]) Let $(X, M, *)$ be a fuzzy metric space then M is a continuous function on $X \times X \times (0, \infty)$.

2. Main Results

In this section, we extend common fixed point theorem of generalized contraction mapping in fuzzy metric spaces. Our work is closely related to [2, 7]. Gregori and Sepena introduced notions of fuzzy contraction mapping and fuzzy contraction sequence as follows:

Definition 2.1 *([7]) Let $(X, M, *)$ be a fuzzy metric space.*

(a) We call the mapping $T : X \to X$ is fuzzy contractive mapping, if there exists $\lambda \in (0, 1)$ such that

$$\frac{1}{M(Tx, Ty, t)} - 1 \leq \lambda \left(\frac{1}{M(x, y, t)} - 1 \right),$$

for each $x, y \in X$ and $t > 0$.

(b) A sequence $\{x_n\}$ is called fuzzy contractive if there exists $\lambda \in (0, 1)$ such that

$$\frac{1}{M(x_n, x_{n+1}, t)} - 1 \leq \lambda \left(\frac{1}{M(x_{n-1}, x_n, t)} - 1 \right),$$

for every $t > 0$, $n \in \mathbb{N}$.

For a family of generalized contraction mapping the following generalize Theorem 4.4 of [7].

Proposition 2.2 *([7]) If sequence $\{x_n\}$ is fuzzy contractive in $(X, M, *)$ then it is G-Cauchy.*

Theorem 2.3 *Let $(X, M, *)$ be a G-complete fuzzy metric space endowed with minimum t-norm and $\{T_\alpha\}_{\alpha \in J}$ be a family of self mappings of X. If there exists a fixed $\beta \in J$ such that for each $\alpha \in J$*

$$\begin{aligned} \frac{1}{M(T_\alpha x, T_\beta y, t)} - 1 \leq{}& \alpha_1 \left(\frac{1}{M(x, y, t)} - 1 \right) + \alpha_2 \left(\frac{1}{M(x, T_\alpha x, t)} - 1 \right) \\ &+ \alpha_3 \left(\frac{1}{M(y, T_\beta y, t)} - 1 \right) + \alpha_4 \left(\frac{1}{M(y, T_\alpha x, 2t)} - 1 \right) \\ &+ \alpha_5 \left(\frac{1}{M(x, T_\beta y, t)} - 1 \right), \qquad (1) \end{aligned}$$

for each $x, y \in X, t > 0$ and for some $0 \leqslant \alpha_5$ and $0 \leqslant \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 < 1$. Then all T_α have a unique common fixed point and if $0 \leqslant \alpha_5 < 1, 0 \leqslant \alpha_2 + \alpha_5 < 1$ then at this point each T_α is continuous.

Proof Let $\alpha \in J$ and $x \in X$ be arbitrary. Consider a sequence, defined inductively by $x_0 = x$ and $x_{2n+1} = T_\alpha x_{2n}, x_{2n+2} = T_\beta x_{2n+1}$ for all $n \geqslant 0$. From (1) we get

$$\begin{aligned}
&\frac{1}{M(x_{2n+1}, x_{2n+2}, t)} - 1 = \frac{1}{M(T_\alpha x_{2n}, T_\beta x_{2n+1}, t)} - 1 \\
&\leq \alpha_1 \left(\frac{1}{M(x_{2n}, x_{2n+1}, t)} - 1\right) + \alpha_2 \left(\frac{1}{M(x_{2n}, x_{2n+1}, t)} - 1\right) \\
&\quad \alpha_3 \left(\frac{1}{M(x_{2n+1}, x_{2n+2}, t)} - 1\right) + \alpha_4 \left(\frac{1}{M(x_{2n}, x_{2n+2}, 2t)} - 1\right) \\
&+ \alpha_5 \left(\frac{1}{M(x_{2n+1}, x_{2n+1}, t)} - 1\right). \qquad (2)
\end{aligned}$$

Since

$$\begin{aligned}
\frac{1}{M(x_{2n}, x_{2n+2}, 2t)} - 1 &\leq \frac{1}{\min\{M(x_{2n}, x_{2n+1}, t), M(x_{2n+1}, x_{2n+2}, t)\}} - 1 \\
&= \max\left\{\frac{1}{M(x_{2n}, x_{2n+1}, t)} - 1, \frac{1}{M(x_{2n+1}, x_{2n+2}, t)} - 1\right\} \\
&\leqslant \left(\frac{1}{M(x_{2n}, x_{2n+1}, t)} - 1\right) \\
&+ \left(\frac{1}{M(x_{2n+1}, x_{2n+2}, t)} - 1\right), \qquad (3)
\end{aligned}$$

combine equations (2) and (3), we get

$$(1 - \alpha_3 - \alpha_4)\left(\frac{1}{M(x_{2n+1}, x_{2n+2}, t)} - 1\right) \leq (\alpha_1 + \alpha_2 + \alpha_4)\left(\frac{1}{M(x_{2n}, x_{2n+1}, t)} - 1\right).$$

Hence,

$$\frac{1}{M(x_{2n+1}, x_{2n+2}, t)} - 1 \leq \lambda \left(\frac{1}{M(x_{2n}, x_{2n+1}, t)} - 1\right),$$

where, by the asumtion, $\lambda = \frac{\alpha_1+\alpha_2+\alpha_4}{1-\alpha_3-\alpha_4}$ belongs to $(0, 1)$. Similarly, we get that

$$\frac{1}{M(x_{2n}, x_{2n+1}, t)} - 1 \leq \lambda \left(\frac{1}{M(x_{2n-1}, x_{2n}, t)} - 1\right).$$

So $\{x_n\}$ is fuzzy contractive, thus, by Proposition 2.2 is G-Cauchy. Since X is G-complete, $\{x_n\}$ converges to u for some $u \in X$. From (1) we have

$$\begin{aligned}
&\frac{1}{M(T_\beta u, x_{2n+1}, t)} - 1 = \frac{1}{M(T_\beta u, T_\alpha x_{2n}, t)} - 1 \\
&\leq \alpha_1 \left(\frac{1}{M(u, x_{2n}, t)} - 1\right) + \alpha_2 \left(\frac{1}{M(u, T_\beta u, t)} - 1\right) \\
&+ \alpha_3 \left(\frac{1}{M(x_{2n}, x_{2n+1}, t)} - 1\right) + \alpha_4 \left(\frac{1}{M(u, x_{2n+1}, 2t)} - 1\right) \\
&+ \alpha_5 \left(\frac{1}{M(x_{2n}, T_\beta u, 2t)} - 1\right).
\end{aligned}$$

Taking the limit as infinity we obtain

$$\frac{1}{M(T_\beta u, u, t)} - 1 \le \alpha_2 \left(\frac{1}{M(u, T_\beta u, t)} - 1 \right).$$

Thus $M(u, Tu, t) = 1$, hence, $T_\beta u = u$. Now we show that u is a fixed point of all $\{T_\alpha \in J\}$. Let $\alpha \in J$. From (1) and Remark *1*, we have

$$\begin{aligned}\frac{1}{M(u, T_\alpha u, t)} - 1 &= \frac{1}{M(T_\beta u, T_\alpha u, t)} - 1 \\ &\le \alpha_2 \left(\frac{1}{M(u, T_\alpha u, t)} - 1 \right) + \alpha_4 \left(\frac{1}{M(u, T_\alpha u, 2t)} - 1 \right) \\ &\le (\alpha_2 + \alpha_4) \left(\frac{1}{M(u, T_\alpha u, t)} - 1 \right).\end{aligned}$$

Hence $T_\alpha u = u$, since α is arbitrary all $\{T_\alpha\}_{\alpha \in J}$ have a common point.

Suppose that v is also a fixed point of T_β. Similar to above, v is a common fixed point of all $\{T_\alpha\}_{\alpha \in J}$. Form (1) we get

$$\frac{1}{M(v, u, t)} - 1 = \frac{1}{M(T_\beta v, T_\alpha u, t)} - 1 \le \alpha_2 \left(\frac{1}{M(u, T_\alpha u, t)} - 1 \right).$$

Thus u is a unique common fixed point of all $\{T_\alpha\}_{\alpha \in J}$. It remains to show each T_α is continuous at u. Let $\{y_n\}$ be a sequence in X such that $y_n \to u$ as $n \to \infty$. From (1) we have

$$\begin{aligned}\frac{1}{M(T_\alpha y_n, T_\alpha u, t)} - 1 &= \frac{1}{M(T_\alpha y_n, T_\beta u, t)} - 1 \\ \le \alpha_1 &\left(\frac{1}{M(y_n, u, t)} - 1 \right) + \alpha_2 \left(\frac{1}{M(y_n, T_\alpha y_n, t)} - 1 \right) \\ + \alpha_4 &\left(\frac{1}{M(y_n, u, 2t)} - 1 \right) + \alpha_5 \left(\frac{1}{M(u, T_\alpha y_n, t)} - 1 \right) \qquad (4)\end{aligned}$$

and similar to (3) we have

$$\frac{1}{M(y_n, T_\alpha y_n, t)} - 1 \le \max \left\{ \left(\frac{1}{M(y_n, u, t/2)} - 1 \right), \left(\frac{1}{M(T_\alpha y_n, u, t/2)} - 1 \right) \right\}. \qquad (5)$$

Combine (4) and (5) we deduce

$$\begin{aligned}&\frac{1}{M(T_\alpha y_n, T_\alpha u, t)} - 1 \le \frac{\alpha_1}{1 - \alpha_5} \left(\frac{1}{M(y_n, u, t)} - 1 \right) \\ &+ \frac{\alpha_4}{1 - \alpha_5} \left(\frac{1}{M(y_n, u, 2t)} - 1 \right) \\ &+ \frac{\alpha_2}{1 - \alpha_5} \max \left\{ \left(\frac{1}{M(y_n, u, t/2)} - 1 \right), \left(\frac{1}{M(T_\alpha y_n, u, t/2)} - 1 \right) \right\}, \qquad (6)\end{aligned}$$

for all $t > 0, n \in \mathbb{N}$. So by (6) and Remark *1* we have

$$\liminf_{n\to+\infty} M(T_\alpha y_n, T_\alpha u, t) \geqslant \frac{1-\alpha_5}{\alpha_2} \limsup_{n\to+\infty} M(T_\alpha y_n, T_\alpha u, t/2)$$
$$\geqslant \frac{1-\alpha_5}{\alpha_2} \limsup_{n\to+\infty} M(T_\alpha y_n, T_\alpha u, t), \tag{7}$$

for all $t > 0$. Thus

$$\lim_{n\to+\infty} M(T_\alpha y_n, T_\alpha u, t) = \lim_{n\to+\infty} M(T_\alpha y_n, T_\alpha u, t/2) = L, \tag{8}$$

exists, for all $t > 0$, and then L equals 1, since in opposite case, applying (6)-(8), one can easily concluded that $\alpha_2 + \alpha_5 \geqslant 1$, contrary to assumption. Thus T_α is continuous at a fixed point. ■

• The mapping in the preceding theorem is called generalized contraction mapping (see [2]). Note that every fuzzy contractive mapping satisfies condition (1).

THEOREM 2.4 *Let $(X, M, *)$ be a complete non-Archimedean fuzzy metric space endowed with minimum t-norm and $\{T_\alpha\}_{\alpha\in J}$ be a family of self mappings of X. If there exists a fixed $\beta \in J$ such that for each $\alpha \in J$*

$$\frac{1}{M(T_\alpha x, T_\beta y, t)} - 1 \leq \alpha_1 \left(\frac{1}{M(x,y,t)} - 1\right) + \alpha_2 \left(\frac{1}{M(x, T_\alpha x, t)} - 1\right)$$
$$+ \alpha_3 \left(\frac{1}{M(y, T_\beta y, t)} - 1\right) + \alpha_4 \left(\frac{1}{M(x, T_\beta y, t)} - 1\right)$$
$$+ \alpha_5 \left(\frac{1}{M(y, T_\alpha x, t)} - 1\right),$$

for each $x, y \in X, t > 0$ and for some $0 < \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$. Then all T_α have a unique common fixed point and at this point each T_α is continuous.

Proof The proof is very similar to Theorem 2.3. In stead of the equation (3) we have

$$\frac{1}{M(x_{n-1}, x_{n+1}, t)} - 1 \leq \frac{1}{\min\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\}} - 1$$
$$= \max\left\{\frac{1}{M(x_{n-1}, x_n, t)} - 1, \frac{1}{M(x_n, x_{n+1}, t)} - 1\right\}.$$

Proceed as the proof of the Theorem 2.3 then we conclude sequence $\{x_n\}$ is fuzzy contractive, thus by [7, Proposition 2.4] and [8, Lemma 2.5], $\{x_n\}$ converges to u for some $u \in X$. Proceed as the proof of the Theorem 2.3. ■

The following provide a converse to Theorem 2.3.

THEOREM 2.5 *Let $(X, M, *)$ be a G-complete fuzzy metric space endowed with minimum t-norm. The following property is equivalent to G-completeness of X:*

If Y is any non empty closed subset of X and $T : Y \to Y$ is any generalized contraction mapping then T has a fixed point in Y.

Proof The sufficient condition follows from Theorem 2.3. Suppose now that the property holds, but $(X, M, *)$ is not complete. Then there exists a Chuchy sequence

$\{x_n\}$ in X which does not converge. We may assume that $M(x_n, x_m, t) < 1$ for all $m \neq n$ and for some $t > 0$. For any $x \in X$ define

$$r(x) = \inf\left\{\frac{1}{M(x_n, x, t)} - 1; x_n \neq x, n = 0, 1, \ldots\right\}.$$

Clearly for all $x \in X$ we have $r(x) > 0$, as $\{x_n\}$ has not a convergent subsequence. Let $\alpha_1 = \alpha_2 = \alpha_3 = 2\alpha_4 = \alpha_5 = 1/8$. We choose a subsequence $\{x_{i_n}\}$ of $\{x_n\}$ as follows. We define inductively a subsequence of positive integer greater than i_{n-1} and such that $\frac{1}{M(x_i, x_k, t)} - 1 \leqslant \alpha_1 r(x_{i_{n-1}})$ for all $i, k \geqslant i_n, n \geqslant 1$. This can done, as $\{x_n\}$ is a Chuchy sequence.

Now define $Tx_{i_n} = x_{i_{n+1}}$ for all n. Then for any $n > m \geqslant 0$ we have

$$\begin{aligned}
\frac{1}{M(Tx_{i_n}, Tx_{i_m}, t)} - 1 &= \frac{1}{M(x_{i_{n+1}}, x_{i_{m+1}}, t)} - 1 \\
&\leq \alpha_1 r(x_{i_m}) \leq \alpha_1\left(\frac{1}{M(x_{i_n}, x_{i_m}, t)} - 1\right) \\
&\leq \alpha_1\left(\frac{1}{M(x_{i_n}, x_{i_m}, t)} - 1\right) + \alpha_2\left(\frac{1}{M(x_{i_n}, x_{i_{n+1}}, t)} - 1\right), \\
&\quad + \alpha_3\left(\frac{1}{M(x_{i_m}, x_{i_{m+1}}, t)} - 1\right) + \alpha_4\left(\frac{1}{M(x_{i_n}, x_{i_{m+1}}, 2t)} - 1\right) \\
&\quad + \alpha_5\left(\frac{1}{M(x_{i_m}, x_{i_{n+1}}, t)} - 1\right) \\
&= \alpha_1\left(\frac{1}{M(x_{i_n}, x_{i_m}, t)} - 1\right) + \alpha_2\left(\frac{1}{M(x_{i_n}, Tx_{i_n}, t)} - 1\right) \\
&\quad + \alpha_3\left(\frac{1}{M(x_{i_m}, Tx_{i_m}, t)} - 1\right) + \alpha_4\left(\frac{1}{M(x_{i_n}, Tx_{i_m}, 2t)} - 1\right) \\
&\quad + \alpha_5\left(\frac{1}{M(x_{i_m}, Tx_{i_n}, t)} - 1\right).
\end{aligned}$$

Thus T is a general contraction mapping on $Y = \{x_{i_n}\}$. Clearly, Y is closed and T has not a fixed point in Y. Thus we get a contradiction. ■

3. Conclusions

In this paper, a theorem on the existence of a common fixed point is proved which characterizes G-completeness of fuzzy metric spaces.

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