

A Regularization Method for Solving a Nonlinear Backward Inverse Heat Conduction Problem Using Discrete Mollification Method

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Abstract. The present essay scrutinizes the application of discrete mollification as a filtering procedure to solve a nonlinear backward inverse heat conduction problem in one dimensional space. These problems are seriously ill-posed. So, we combine discrete mollification and space marching method to address the ill-posedness of the proposed problem. Moreover, a proof of stability and convergence of the aforementioned algorithm is provided. Finally, the results of this paper have been illustrated by some numerical examples.

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1. Introduction

The inverse heat conduction problems are widely practiced in many branches of physics, science and engineering. In most of the problems related to conduction, temperature distribution becomes obvious at the last moment of the process. The question is that how it is possible to obtain the temperature distribution at the beginning moment using this data. These types of problems are called backward

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inverse heat conduction problems (BIHCP) or final boundary value problems. BIHCP is practiced in a wide range of applied areas including thermophysics, mechanics of continuous media, contaminant transport, medical imaging, geophysics and explorations and, etc; see e.g. [15-20]. These problems are seriously ill-posed, it means that their solution (if it exists) does not continuously depend on the data. In this case, some minor and partial changes in the data are directed into major and remarkable changes in the computed solution. Holistically, the above mentioned problems cannot be solved by classic numerical methods. Thus, the efficient and effective regularization procedures are employed to solve them [8,13].

Within the last few years, various numerical and analytical methods have been introduced to solve BIHCP such as regularization method based on generalization of the BIHCP [9], modified integral equation method [10], variational approach [11], modified Tikhonov regularization method [14], Euler and Crank-Nicolson methods [22] and Shannon wavelet regularization method [23]. However, most of the essays in this issue have been limited to the linear backward problems and just a few studies have been done on the nonlinear backward inverse heat conduction problems (NBIHCP).

In this article, a NBIHCP in one dimensional space is considered. This research aims at proposing a numerical algorithm based on discrete mollification and space marching methods and assessing necessary conditions for creating its stability and convergence. Mollification method is recognized as a reliable regularization method based on convolution that has been widely applied to solve many ill-posed problems [5,12]. The idea of this method is very simple [21]: if the data of the problem are not clear and only an approximate amount of data is accessible, it is recommended to find out a sequence of mollification operators to map improper data into well-posed classes of the problem (mollify the improper data). Consequently, the intended problem will be a well-posed one.

This paper is organized as the following: The second section represents the backward inverse heat conduction problem. The third section reviews the discrete mollification method. In the fourth one, we regularize the intended backward inverse conduction problem. The regularized problem is solved in the fifth section. The sixth section considers the stability and convergence proof of the space marching numerical algorithm. Eventually, some numerical examples are defined in the last section.

2. The Mathematical Formulation of the Problem

Let us consider a nonlinear initial boundary value heat conduction problem which is governed by

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial}{\partial x} \{ (a(x) + b(x)u^2(x,t)) \frac{\partial u(x,t)}{\partial x} \} = f(x,t), \quad 0 < x < 1, \quad 0 < t < T, \quad (1)$$

$$u(x,0) = \psi(x), \quad 0 \leq x \leq 1, \quad (2)$$

$$u(0,t) = g_1(t), \quad 0 \leq t \leq T, \quad (3)$$

$$u(1,t) = g_2(t), \quad 0 \leq t \leq T, \quad (4)$$

where $u(x,t)$ is the unknown temperature in the range of $(0,1) \times [0,T]$. Furthermore, $a(x) + b(x)u^2(x,t)$ identified as the diffusion coefficient is a positive function. Moreover, source term $f(x,t)$, the initial condition $\psi(x)$ and the boundary conditions $g_1(t)$, $g_2(t)$ are known functions in all their domains. T is also given. This

problem is called direct heat conduction problem which can be solved by classic numerical methods applied to solve PDEs. The existence and uniqueness of solution of problem (1)-(4) in homogeneous case are investigated in [24].

Corresponding to this problem, we consider an inverse problem in which the initial condition $\psi(x)$ and the boundary condition $g_2(t)$ are unknown. We suppose that there is an extra set of the measured temperature at the final time $t = T$ and the heat flux at the boundary of $x = 0$:

$$u(x, T) = \varphi(x), \quad 0 \leq x \leq 1, \quad (5)$$

$$\frac{\partial u(0, t)}{\partial x} = 0, \quad 0 \leq t \leq T. \quad (6)$$

Equations (1)-(4) along with overspecified conditions of (5) and (6) are called a NBIHCP.

In the following, a stable and convergent numerical algorithm based on discrete mollification and space marching will be introduced to find the solution of problem (1)-(6). Now, we suppose that functions $\varphi(x)$ and $g_1(t)$ are not known exactly and we only have an approximate amount of these functions presented as $\varphi^\varepsilon(x)$ and $g_1^\varepsilon(t)$, respectively. In addition, these approximate functions are satisfied in the following conditions:

$$\|\varphi^\varepsilon(x) - \varphi(x)\|_\infty \leq \varepsilon,$$

$$\|g_1^\varepsilon(x) - g_1(x)\|_\infty \leq \varepsilon.$$

According to the fact that there are perturbation in the problem's data and the proposed problem is ill-posed, we first regularize the inverse problem by discrete mollification method.

In the next part, we precisely review the discrete mollification method.

3. A Review on the Discrete Mollification Method

Discrete mollification method is a filtering procedure, based on convolution, which has proven to be convenient for the regularization of a variety of ill-posed problems and for stabilization of explicit space marching algorithms in the solution of PDEs [3]. In this section from [6,7], the basic idea of discrete mollification method is introduced. For more information about this method see [1].

Let $G = \{g(x_j) = g_j\}_{j=1}^M$ be a discrete function defined on $K = \{x_j, j = 1, \dots, M\} \subset [0, 1]$ satisfying

$$0 \leq x_1 < x_2 < \dots < x_{M-1} < x_M \leq 1.$$

Set

$$s_j = \begin{cases} 0, & j = 0 \\ \frac{1}{2}(x_j + x_{j+1}), & j = 1, \dots, M-1 \\ 1, & j = M. \end{cases}$$

Let $p > 0$ is given. Then for any $x \in I_\delta = [p\delta, 1 - p\delta]$ we define discrete mollification of G as follows

$$J_\delta G(x) = \sum_{j=1}^M \left(\int_{s_{j-1}}^{s_j} \rho_{\delta,p}(x-s) ds \right) g_j,$$

where

$$\rho_{\delta,p}(x) = \begin{cases} A_p \delta^{-1} \exp(-\frac{x^2}{\delta^2}), & |x| \leq p\delta \\ 0, & |x| > p\delta, \end{cases}$$

such that $A_p = (\int_{-p}^p \exp(-s^2) ds)^{-1}$. We usually take $p=3$ and the radius of mollification, δ is selected automatically by the GCV method (see more [1]). We note that

$$\sum_{j=1}^M \int_{s_{j-1}}^{s_j} \rho_{\delta,p}(x-s) ds = \int_{-p\delta}^{p\delta} \rho_{\delta,p}(s) ds = 1.$$

Set

$$\Delta x = \max_{1 \leq j \leq M-1} |x_{j+1} - x_j|.$$

In sequence, we will introduce the main properties relating discrete mollification method (see more [1,2]).

Theorem 1. ([1])

1. Let $g(x) \in C^{0,1}(R^1)$ and $G = \{g(x_j) = g_j\}_{j=1}^M$ be the discrete version of g and let $G^\varepsilon = \{g_j^\varepsilon\}_{j=1}^M$ be the perturbed discrete version of g satisfying $\|G - G^\varepsilon\|_{\infty,K} \leq \varepsilon$. Then there exists a constant C , independent of δ , such that

$$\|J_\delta G^\varepsilon - J_\delta g\|_\infty \leq C(\varepsilon + \Delta x).$$

2. If $g'(x) \in C^{0,1}(R^1)$, let $G = \{g(x_j) = g_j\}_{j=1}^M$ and $G^\varepsilon = \{g_j^\varepsilon\}_{j=1}^M$ satisfying $\|G - G^\varepsilon\|_{\infty,K} \leq \varepsilon$, then

$$\|D(J_\delta G^\varepsilon) - (J_\delta g)'\|_\infty \leq \frac{C}{\delta}(\varepsilon + \Delta x) + C_\delta(\Delta x)^2.$$

3. Suppose that $G = \{g(x_j) = g_j\}_{j=1}^M$ be the discrete function defined on K , and D_0^δ be a differentiation operator defined by $D_0^\delta(G) = D(J_\delta G)(x)$ then

$$\|D_0^\delta(G)\|_{\infty,K} \leq \frac{C}{\delta} \|G\|_{\infty,K}.$$

In order to compute $J_\delta G(x)$ throughout the domain $[0, 1]$, we have to extend discrete data function g to a bigger interval $I_{\delta'} = [-p\delta, 1 + p\delta]$ or confine this function to the $I_\delta = [p\delta, 1 - p\delta]$. In this essay, the first approach described in reference [1] is applied. An optimizing process is practiced to calculate the extension function of g in the intervals of $[-p\delta, 0]$ and $[1, 1 + p\delta]$. This process is introduced by Mejia in the reference [4].

4. The Regularized Problem

The regularized problem is described as

$$\frac{\partial v(x, t)}{\partial t} - \frac{\partial}{\partial x} \left\{ (a(x) + b(x)v^2(x, t)) \frac{\partial v(x, t)}{\partial x} \right\} = f(x, t), \quad 0 < x < 1, \quad 0 < t < T, \quad (7)$$

$$v(x, 0) = J_{\delta} \psi(x), \quad 0 \leq x \leq 1, \quad (8)$$

$$v(0, t) = J_{\delta_1} g_1(t), \quad 0 \leq t \leq T, \quad (9)$$

$$v(1, t) = J_{\delta'} g_2(t), \quad 0 \leq t \leq T, \quad (10)$$

with overspecified conditions

$$v(x, T) = J_{\delta_2} \varphi(x), \quad 0 \leq x \leq 1, \quad (11)$$

$$\frac{\partial v(0, t)}{\partial x} = 0, \quad 0 \leq t \leq T. \quad (12)$$

The aim of this problem is to find $v(x, t)$ satisfying (7)-(12).

5. The space marching algorithm

Now, we solve regularized problem by the means of space marching method and determine $v(x, t) \in ([0, 1] \times [0, T])$. For this purpose, we first discretize domain $[0, 1] \times [0, T]$ with the mesh points

$$x_j = jh, \quad j = 0, \dots, M,$$

and

$$t_n = nk, \quad n = 0, \dots, N,$$

which $h = 1/M$, $k = T/N$ are the space and time discretization parameters. Let the numerical approximations of functions $v(jh, nk)$, $v_t(jh, nk)$, $v_x(jh, nk)$, $f(jh, nk)$, $a(jh)$ and $b(jh)$ are indicated as U_j^n , W_j^n , R_j^n , f_j^n , a_j and b_j , respectively.

The space marching scheme for (7)-(12) is

$$U_{j+1}^n = U_j^n + hR_j^n, \quad (13)$$

$$R_{j+1}^n = \frac{1}{a_{j+1} + b_{j+1}(U_{j+1}^n)^2} \{ (a_j + b_j(U_j^n)^2) R_j^n + h(W_j^n - f_j^n) \}, \quad (14)$$

$$W_{j+1}^n = W_j^n + h(D_0)_t (J_{\delta_3^j} R_j^n), \quad (15)$$

where D_0 is the centered difference operator denoting by

$$D_0 f(t) = \frac{f(t + \Delta t) - f(t - \Delta t)}{2\Delta t}.$$

The algorithm of this scheme is as follows

1. Choose the radii of mollification, δ_1 and δ_2 using GCV method.

2. Compute mollification of $g_1^\varepsilon(nk)$ and $\varphi^\varepsilon(jh)$ with respect to t and x , respectively.

Put

$$\begin{aligned} U_0^n &= J_{\delta_1} g_1^\varepsilon(nk), n = 0, \dots, N, \\ U_j^N &= J_{\delta_2} \varphi^\varepsilon(jh), j = 1, \dots, M, \\ R_0^n &= 0, n = 0, \dots, N. \end{aligned}$$

3. Compute $D_0(J_{\delta_1} g_1^\varepsilon(nk))$ and put

$$W_0^n = D_0(J_{\delta_1} g_1^\varepsilon(nk)), n = 0, \dots, N.$$

4. Set $j = 0$ and do while $j \leq M - 1$

$$\begin{aligned} U_{j+1}^n &= U_j^n + hR_j^n, (n \neq N), \\ R_{j+1}^n &= \frac{1}{a_{j+1} + b_{j+1}(U_{j+1}^n)^2} \{ (a_j + b_j(U_j^n)^2)R_j^n + h(W_j^n - f_j^n) \}, \\ W_{j+1}^n &= W_j^n + h(D_0)_t(J_{\delta_3^j} R_j^n). \end{aligned}$$

This algorithm is applied to solve some examples in section 6.

In order to analyze stability and convergence of the numerical scheme, we assume $f(x, t) \in C([0, 1] \times [0, T])$, $a(x), b(x) \in C^1[0, 1]$ and also $u(x, t) \in C^2([0, 1] \times [0, T])$.

6. Stability and convergence of the scheme

In this section, we establish stability and convergence of the space marching scheme (13)-(15). Sequentially, we bring a hypothesis that will be needed.

Hypothesis 1. Consider problem (7)-(12) and suppose that

1. For all $(x, t) \in ([0, 1] \times [0, T])$, $a(x) + b(x)u^2(x, t)$ is increasing with respect to x .
2. For all $j = 0, 1, \dots, M$ and $n = 0, 1, \dots, N$, $a_j + b_j(U_j^n)^2$ is increasing with respect to j .

From now on, we use the notation

$$|Y_j| = \max_n |Y_j^n|.$$

Theorem 2. (Stability theorem) Suppose that hypothesis 1 is held, then there exists constant M_1 , such that

$$\max\{|U_M|, |R_M|, |W_M|, M_f\} \leq \exp\{M_1\} \max\{|U_0|, |R_0|, |W_0|, M_f\}.$$

Proof From (13) and (14), we obtain

$$|U_{j+1}^n| \leq (1 + h) \max\{|U_j^n|, |R_j^n|\}. \quad (16)$$

Using theorem 1 and Eqn. (15), we achieve

$$|W_{j+1}^n| \leq (1 + h \frac{C}{|\delta|_{-\infty}}) \max\{|R_j^n|, |W_j^n|\}, \quad (17)$$

where C is a constant which is independent of δ and $|\delta|_{-\infty} = \min_j(\delta_3^j)$.

Hypothesis 1 and Eqn. (14) are directed to

$$|R_{j+1}^n| \leq (|R_j^n| + hd_{j+1}(|W_j^n| + |f_j^n|)),$$

consequently

$$|R_{j+1}^n| \leq (1 + dh) \max\{|R_j^n|, |W_j^n|, M_f\}, \quad (18)$$

where

$$d_{j+1} = \min_n \{a_{j+1} + b_{j+1} |U_{j+1}^n|^2\},$$

$$d = \max_j d_j,$$

and

$$M_f = \max_{(x,t) \in ([0,1] \times [0,T])} |f(x,t)|,$$

following (16)-(18)

$$\max\{|U_{j+1}|, |R_{j+1}|, |W_{j+1}|, M_f\} \leq (1 + hM_1) \max\{|U_j|, |R_j|, |W_j|, M_f\},$$

such that

$$M_1 = \max\{1, \frac{C}{|\delta|_{-\infty}}, d\}.$$

after M iteration of the last inequality, we obtain

$$\max\{|U_M|, |R_M|, |W_M|, M_f\} \leq (1 + hM_1)^M \max\{|U_0|, |R_0|, |W_0|, M_f\},$$

thus

$$\max\{|U_M|, |R_M|, |W_M|, M_f\} \leq \exp(M_1) \max\{|U_0|, |R_0|, |W_0|, M_f\}.$$

So, the space marching scheme (13)-(15) is stable and proof is complete for fixed M_1 .

Theorem 3. (Convergence theorem) For fixed δ , as h , k and ε tend to zero then the numerical scheme (13)-(15) converge to the mollified exact solution.

Proof Let ΔU_j^n , ΔR_j^n and ΔW_j^n be the discrete error functions defined by

$$\begin{aligned} \Delta U_j^n &= U_j^n - v(jh, nk), \\ \Delta R_j^n &= R_j^n - v_x(jh, nk), \\ \Delta W_j^n &= W_j^n - v_t(jh, nk), \end{aligned}$$

then, we have

$$\begin{aligned} \Delta U_{j+1}^n &= U_{j+1}^n - v((j+1)h, nk) \\ &= \Delta U_j^n + (U_{j+1}^n - U_j^n) - (v((j+1)h, nk) - v(jh, nk)) \\ &= \Delta U_j^n + h(R_j^n - v_x(jh, nk)) + O(h^2) \\ &= \Delta U_j^n + h\Delta R_j^n + O(h^2), \end{aligned} \quad (19)$$

$$\begin{aligned}
\Delta W_{j+1}^n &= W_{j+1}^n - v_t((j+1)h, nk) \\
&= \Delta W_j^n + (W_{j+1}^n - W_j^n) - (v_t((j+1)h, nk) - v_t(jh, nk)) \quad (20) \\
&= \Delta W_j^n + h(D_0(J_{\delta_3^j} R_j^n) - v_{xt}(jh, nk)) + O(h^2),
\end{aligned}$$

and

$$\begin{aligned}
\Delta R_{j+1}^n &= R_{j+1}^n - v_x((j+1)h, nk) \\
&= \Delta R_j^n + (R_{j+1}^n - R_j^n) - (v_x((j+1)h, nk) - v_x(jh, nk)).
\end{aligned}$$

Now, through applying Taylor series and using (14) we get

$$\begin{aligned}
\Delta R_{j+1}^n &= \Delta R_j^n + \frac{1}{a_{j+1} + b_{j+1}(U_{j+1}^n)^2} \{(a_j + b_j(U_j^n)^2)R_j^n + h(W_j^n - f_j^n)\} - R_j^n \\
&\quad - hv_{xx}(jh, nk) + O(h^2). \quad (21)
\end{aligned}$$

From Eqn. (19), it is obtained that

$$|\Delta U_{j+1}^n| \leq |\Delta U_j^n| + h |\Delta R_j^n| + O(h^2). \quad (22)$$

Due to the theorem 1 and eqn. (20), it is concluded that

$$|\Delta W_{j+1}^n| \leq |\Delta W_j^n| + h(C \frac{|\Delta R_j^n|}{|\delta|_{-\infty}} + C_\delta k^2) + O(h^2), \quad (23)$$

which C and C_δ are constants.

Finally, Eqn. (21) yields

$$\begin{aligned}
|\Delta R_{j+1}^n| &\leq |\Delta R_j^n| + |R_j^n| \left| \frac{a_j + b_j(U_j^n)^2}{a_{j+1} + b_{j+1}(U_{j+1}^n)^2} - 1 \right| \\
&\quad + h \frac{1}{|a_{j+1} + b_{j+1}(U_{j+1}^n)^2|} (|W_j^n| + |f_j^n|) + h |v_{xx}(jh, nk)| + O(h^2) \\
&\leq |\Delta R_j^n| + l_1 + hl_2 + O(h^2), \quad (24)
\end{aligned}$$

where

$$l_1 = \sup_{j,n} |R_j^n| \left| \frac{a_j + b_j(U_j^n)^2}{a_{j+1} + b_{j+1}(U_{j+1}^n)^2} - 1 \right|,$$

$$l_2 = \sup_{j,n} \left\{ \frac{1}{|a_{j+1} + b_{j+1}(U_{j+1}^n)^2|} (|W_j^n| + |f_j^n|) + |v_{xx}(jh, nk)| \right\}.$$

Set

$$\Delta_j = \max\{|\Delta U_j^n|, |\Delta R_j^n|, |\Delta W_j^n|\}.$$

From Eqns. (22)-(24), we result that

$$\Delta_{j+1} \leq \Delta_j(1 + hl) + hl' + l_1 + O(h^2),$$

which

$$l = \max\left\{1, \frac{C}{|\delta|_{-\infty}}\right\},$$

and

$$l' = l_2 + \frac{Ck}{|\delta|_{-\infty}} + C_\delta k^2.$$

Therefore, after M iteration, we derive

$$\Delta_M \leq \Delta_0(1 + hl)^M + (hl' + l_1)(1 + hl)^{M-1} + \dots + (hl' + l_1)(1 + hl) + (hl' + l_1). \quad (25)$$

Theorem 1 is directed to the following inequality

$$\begin{aligned} |\Delta U_0^n| &\leq C(\varepsilon + k), \\ |\Delta Q_0^n| &\leq C(\varepsilon + k), \\ |\Delta W_0^n| &\leq \frac{C}{|\delta|_{-\infty}}(\varepsilon + k) + C_\delta k^2, \end{aligned}$$

so as ε and k tend to zero, $\Delta_0 \rightarrow 0$.

Then according to above explanation and stability of the scheme (13)-(15) as $h, k \rightarrow 0$, we see that l_1 tend to zero.

Consequently, as ε, h and k tend to zero the right hand side of (25) tend to zero and convergence of the scheme (13)-(15) readily follows.

7. Numerical examples

In this section, we present two numerical examples to demonstrate the effectiveness and stability of our proposed method. Stability of the method with respect to noise in the data is investigated using noisy data. The noisy discrete data functions are generated by adding a random perturbation to the exact data functions. The radii of mollification are chosen automatically by the GCV method. For checking the accuracy of our algorithm, we use relative weighted l^2 -norm. In these examples, we take $T = 1$. To verify the convergence rate, we use the following definition

$$CO = \log_2 \frac{\|\psi - \psi^{2\varepsilon}\|}{\|\psi - \psi^\varepsilon\|}.$$

We apply Mathematica 10.3.1 software for computation.

Example 1 It is easy to observe that

$$a(x) = 0.01 + x^2 e^x,$$

$$b(x) = 1 + x^2,$$

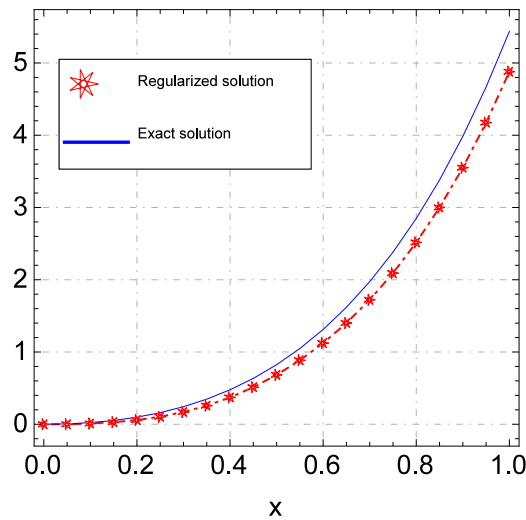


Figure 1. A comparison between the exact and regularized solutions for $\psi(x)$ with $\varepsilon = 0.001$ and $h = k = \frac{1}{20}$ for Example 1.

are satisfied in problem (1)-(6) with the boundary conditions

$$u(0, t) = u_x(0, t) = 0,$$

and final data

$$u(x, 1) = x^2 e^x.$$

$f(x, t)$ is taken so that the exact solution is $u(x, t) = (2 - t)x^2 e^x$.

In order to investigate the impact of the "inverse crime", we consider the following two cases for solving the proposed problem [25].

Case 1: We take $u(x, 1) = x^2 e^x$ and $u_x(0, t) = 0$ and solve this problem by the space marching algorithm with discrete mollification. Then, we carry out the numerical results in Figs. 1-3 for $\psi(x)$ and $g_2(t)$. Table 1 illustrates relative l_2 errors for computed u with different noise levels $\varepsilon = 0.001, 0.01$ and 0.1 . In Table 2, we show that the order of convergence of the proposed method for two noise levels $\varepsilon = 0.05, 0.1$ is about 1 as we expected.

Case 2: We first solve a direct problem to obtain the input data $u(x, 1)$ and $\frac{\partial u(0, t)}{\partial x}$ then solve the inverse problem using the proposed method [25]. Errors of the method are shown in Table 3.

From Table 1 and 3, we can see that at fix noise level ε , the accuracy of our algorithm will be increased by decreasing h and we do not observe the impact of the "inverse crime". Figs. 1-3 reveal the efficiency of discrete mollification method as a regularization procedure. Holistically, we can see as h declines, the accuracy of approximated solutions will enhance.

Example 2 Consider the problem (1)-(6) with the following exact data functions

$$a(x) = 0.001 + x \sin(x),$$

$$b(x) = 1 + x,$$

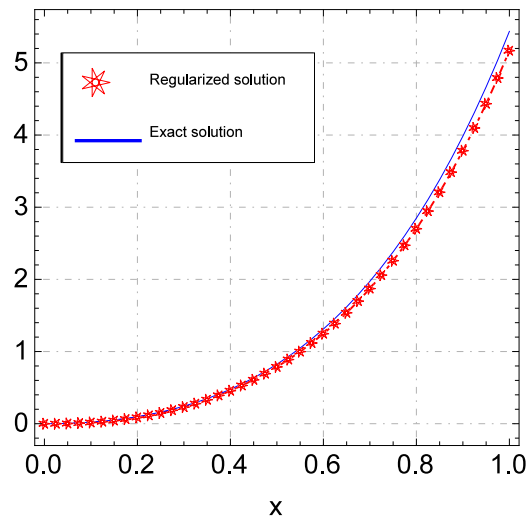


Figure 2. A comparison between the exact and regularized solutions for $\psi(x)$ with $\varepsilon = 0.001$ and $h = k = \frac{1}{40}$ for Example 1.

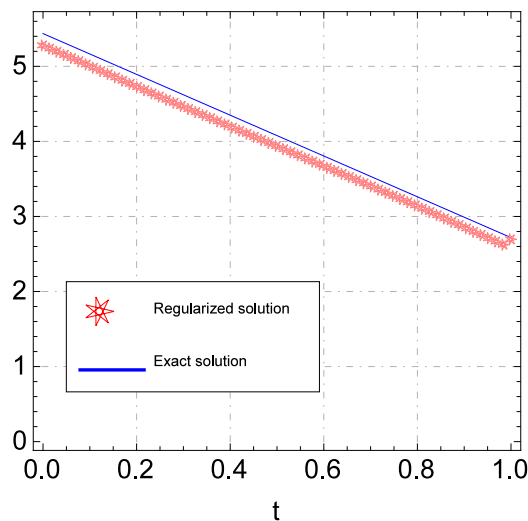


Figure 3. A comparison between the exact and regularized solutions for $g_2(t)$ with $\varepsilon = 0.001$ and $h = k = \frac{1}{64}$ for Example 1.

with the boundary conditions

$$u(0, t) = u_x(0, t) = 0,$$

and final data

$$u(x, 1) = (2 + e)x^3(1 + \sin(x)).$$

$f(x, t)$ is taken so that the exact solution is $u(x, t) = (2 + e^{t^2})x^3(1 + \sin(x))$.

The relative l_2 errors for computed u are listed in Table 4 with three noise levels $\varepsilon = 0.001, 0.01$ and 0.1 . This table illustrates that at fix noise level ε , as h decrease the accuracy of algorithm will improve. In Table 5, we show that the order of convergence of the proposed method for two noise levels $\varepsilon = 0.05, 0.1$ is about 1 as we expected. Conclusively, the comparison between the exact solution and its

Table 1. Relative l^2 error norms for regularized solution u for Example 1, Case 1.

ε	h	k	Relative l^2 error for u
0.001	2^{-4}	2^{-4}	0.1570810
0.001	2^{-4}	2^{-5}	0.1585340
0.001	2^{-5}	2^{-5}	0.0776185
0.001	2^{-5}	2^{-6}	0.0809734
0.001	2^{-5}	2^{-7}	0.0812005
0.001	2^{-6}	2^{-6}	0.0417975
0.001	2^{-6}	2^{-7}	0.0419530
0.001	2^{-7}	2^{-7}	0.0234503
0.01	2^{-4}	2^{-4}	0.1574020
0.01	2^{-4}	2^{-5}	0.1594970
0.01	2^{-5}	2^{-5}	0.0783974
0.01	2^{-5}	2^{-6}	0.0825739
0.01	2^{-5}	2^{-7}	0.0836425
0.01	2^{-6}	2^{-6}	0.0447928
0.01	2^{-6}	2^{-7}	0.0451941
0.01	2^{-7}	2^{-7}	0.0273820
0.1	2^{-4}	2^{-4}	0.1805120
0.1	2^{-4}	2^{-5}	0.1866360
0.1	2^{-5}	2^{-5}	0.1168900
0.1	2^{-5}	2^{-6}	0.1146340
0.1	2^{-5}	2^{-7}	0.1207490
0.1	2^{-6}	2^{-6}	0.0854462
0.1	2^{-6}	2^{-7}	0.0964583
0.1	2^{-7}	2^{-7}	0.0814588

Table 2. The rate of Convergence for Example 1.

ε	h	k	CO
0.05	30	30	0.754962
0.05	35	35	0.875326
0.05	40	40	0.907374
0.1	30	30	0.699285
0.1	35	35	0.854781
0.1	40	40	0.912398

Table 3. Relative l^2 error norms for regularized solution u for Example 1, Case 2.

ε	h	k	Relative l^2 error for u
0.001	2^{-4}	2^{-4}	0.5932642
0.001	2^{-5}	2^{-5}	0.4085852
0.001	2^{-5}	2^{-6}	0.4839570
0.001	2^{-5}	2^{-7}	0.4563291
0.01	2^{-4}	2^{-4}	0.6328715
0.01	2^{-5}	2^{-5}	0.4358147
0.01	2^{-5}	2^{-6}	0.4724916
0.01	2^{-5}	2^{-7}	0.4787219

regularized solution with discrete mollification method for $\psi(x)$ and $g_2(t)$ are illustrated in Figs. 4-6. These figures demonstrate effectiveness of discrete mollification method.

8. Conclusion

In this paper, a nonlinear backward inverse heat conduction problem in one dimensional space is considered. A numerical algorithm based on discrete mollification

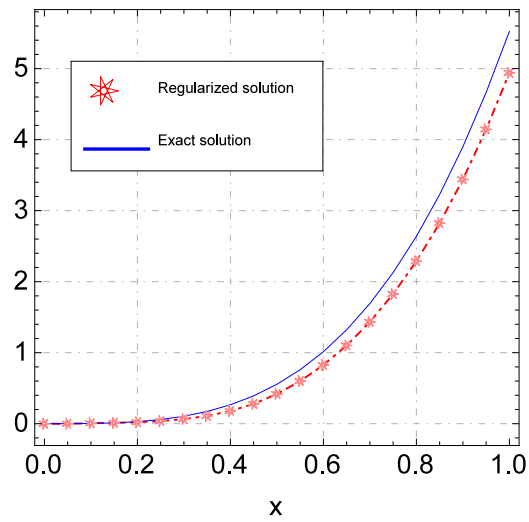


Figure 4. A comparison between the exact and regularized solutions for $\psi(x)$ with $\varepsilon = 0.001$ and $h = k = \frac{1}{20}$ for Example 2.

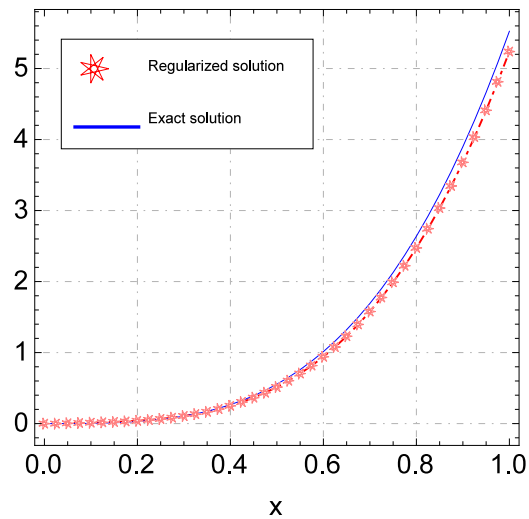


Figure 5. A comparison between the exact and regularized solutions for $\psi(x)$ with $\varepsilon = 0.001$ and $h = k = \frac{1}{40}$ for Example 2.

and space marching methods is proposed and the stability and convergence of the aforementioned algorithm are provided. Numerical results demonstrate effectiveness of discrete mollification method as a reliable method for solving the mentioned problem.

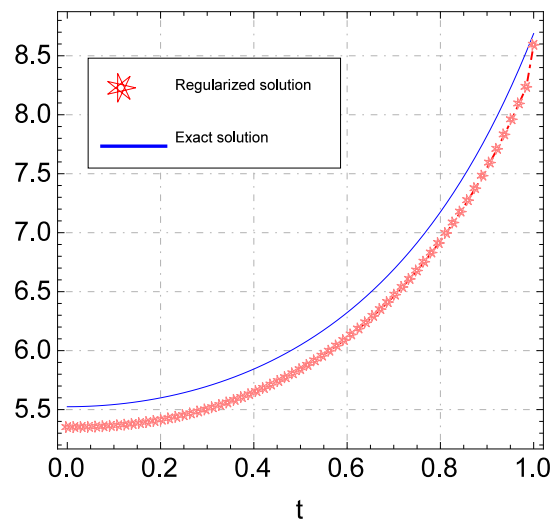


Figure 6. A comparison between the exact and regularized solutions for $g_2(t)$ with $\varepsilon = 0.001$ and $h = k = \frac{1}{64}$ for Example 2.

Table 4. Relative l^2 error norms for regularized solution for Example 2.

ε	h	k	Relative l^2 error for u
0.001	2^{-4}	2^{-4}	0.1491700
0.001	2^{-4}	2^{-5}	0.1529740
0.001	2^{-5}	2^{-5}	0.0783948
0.001	2^{-5}	2^{-6}	0.0794999
0.001	2^{-5}	2^{-7}	0.0800352
0.001	2^{-6}	2^{-6}	0.0408504
0.001	2^{-6}	2^{-7}	0.0410051
0.001	2^{-7}	2^{-7}	0.0215161
0.01	2^{-4}	2^{-4}	0.1497810
0.01	2^{-4}	2^{-5}	0.1538540
0.01	2^{-5}	2^{-5}	0.0808071
0.01	2^{-5}	2^{-6}	0.0817251
0.01	2^{-5}	2^{-7}	0.0826516
0.01	2^{-6}	2^{-6}	0.0444123
0.01	2^{-6}	2^{-7}	0.0446082
0.01	2^{-7}	2^{-7}	0.0266623
0.1	2^{-4}	2^{-4}	0.1710070
0.1	2^{-4}	2^{-5}	0.1775330
0.1	2^{-5}	2^{-5}	0.1062610
0.1	2^{-5}	2^{-6}	0.1090910
0.1	2^{-5}	2^{-7}	0.1118110
0.1	2^{-6}	2^{-6}	0.0716435
0.1	2^{-6}	2^{-7}	0.0732865
0.1	2^{-7}	2^{-7}	0.0667013

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Table 5. The rate of Convergence for Example 2.

ε	h	k	CO
0.05	30	30	0.805462
0.05	35	35	0.889316
0.05	40	40	0.927374
0.1	30	30	0.793176
0.1	35	35	0.864463
0.1	40	40	0.904948

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