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An Efficient Method to Solve the Mathematical Model of HIV Infection for CD8$^+$ T-Cells

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Abstract. In this paper, the mathematical model of HIV infection for CD8$^+$ T-cells is illustrated. The homotopy analysis method and the Laplace transformations are combined for solving this model. Also, the convergence theorem is proved to demonstrate the abilities of presented method for solving non-linear mathematical models. The numerical results for $N = 5, 10$ are presented. Several $\hbar$-curves are plotted to show the convergence regions of solutions. The plots of residual error functions indicate the precision of presented method.

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1. Introduction

Human Immunodeficiency Virus (HIV) is one of the most dangerous viruses in the world that leads to Acquired Immunodeficiency Syndrome (AIDS). This virous involves the ribonucleic acid (RNA) instead of the deoxyribonucleic acid (DNA) and finally the HIV mechanism can be completed during 10-15 years [12]. In 1980,

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the first case of HIV infection was reported. According to the recent enumeration, more than 35 million people have been died by HIV virous and more than 37 million people carry this virous in their body and they are living as a menace on the world. Also, they can transmit this threat by having unprotected sex, forwarding from mother to child and other ways [33, 36, 46, 51].

In last decades, many mathematical models have been presented to identify the behavior of natural and artificial phenomena such as mathematical model of HIV infection [34, 40, 54], model of Malaria viruses [53], model of computer viruses [38, 39, 44] and many other models [13]. Also, these models have been solved by many numerical or semi-analytical methods.

The HAM is among of the semi-analytical methods which has been presented by Liao [28–32]. In this method, we have an operator, parameters and functions that we have freedom to choose them. Selecting prepare parameters can lead to find the solution of problem faster and more accurate than other semi-analytical methods. In last decade, many authors applied the HAM for solving mathematical and bio-mathematical problems such as model of computer viruses [44], model of HIV infection for CD4$^+$ T-cells [40], ill-posed problems [6] and others [17–23]. Moreover, recently in [42] we applied the CESTAC method [7, 8, 41] and the CADNA library [11, 43] based on the stochastic arithmetic to find the optimal step, the optimal error and the optimal value of convergence control parameter of the HAM.

In some schemes, by combining the HAM by other methods or operators we can construct new methods such as combining the HAM and Laplace transformations (HATM) [9, 27, 40, 45], optimal homotopy analysis method [37], discrete homotopy analysis method [55], predictor homotopy analysis method [1, 50], homotopy analysis Sumudu transform method [26] and many others [2, 10, 47, 49].

The aim of this paper is to present the HATM to solve the following non-linear bio-mathematical model [36]

$$
\begin{aligned}
\frac{dT(t)}{dt} &= \lambda_T - \mu_T T(t) - \chi T(t)V(t)\\
\frac{dI(t)}{dt} &= \chi T(t)V(t) - \mu_I I(t) - \alpha I(t)Z_a(t),\\
\frac{dV(t)}{dt} &= \epsilon_V \mu_I I(t) - \mu_V V(t),\\
\frac{dZ(t)}{dt} &= \lambda_Z - \mu_Z Z(t) - \beta Z(t)I(t),\\
\frac{dZ_a(t)}{dt} &= \beta Z(t)I(t) - \mu_{Z_a} Z_a(t),
\end{aligned}
\tag{1}
$$

where $T(t)$ and $I(t)$ show the condensation of the susceptible and infected CD4$^+$ T-cells at any time t, $V(t)$ is the condensation of infectious HIV viruses and finally $Z(t)$ and $Z_a(t)$ are the condensation of the CD8$^+$ T-cells and population of the activated CD8$^+$ T-cells at any time t. List of parameters and their values are presented in Table 1 [3, 4, 33, 46, 51, 52]. Moreover, in Figures 1 and 2 the life cycle of HIV infection and its model on CD8$^+$ T-cells are demonstrated [36].

The HATM obtains by combining the HAM with Laplace transformations. Recently, the HATM has been applied to solve the various problems such as solving singular problems [45], fractional modeling for BBM-Burger equation [24], Klein-Gordon equations [25], fractional diffusion problem [5], partial differential equations [35], fuzzy problems [9, 48] and others [14–16].

This research is organized in the following form: Section 2 is the main idea for

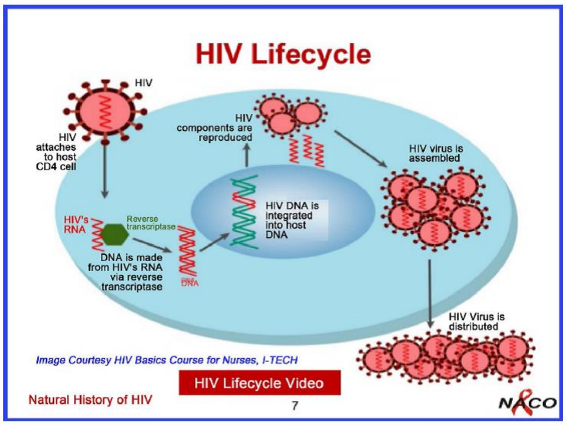

Figure 1. HIV life cycle.

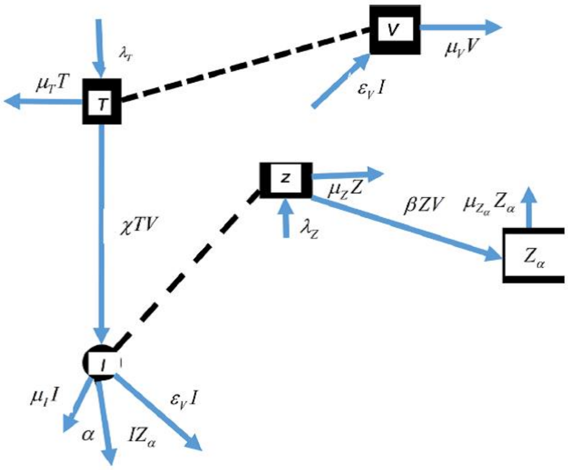

Figure 2. Diagram of HIV infection model of CD8$^+$ T-cells.

solving the non-linear bio-mathematical model 1. The convergence theorem for solving presented model is illustrated in Section 3. In Section 4, the numerical results for $N = 5, 10$ are presented. Also, several $\hbar$-curves are demonstrated to show the convergence regions of this problem. Furthermore, the plots of residual error functions are presented to show the precision of method. Finally, Section 5 is conclusion.

Table 1. List of parameters and their values.

Parameters	Meaning	Values
λ_T	Rate of recruiting the susceptible CD4$^+$ T-cells per unit time.	10 cell/mm^3/day
μ_T	Rate of decaying for susceptible CD4$^+$ T-cells.	0.01 day^{-1}
χ	Rate of infecting for CD4$^+$ T-cells by the virus.	0.000024 mm^3 vir^{-1} day^{-1}
μ_I	Rate of the natural death for infected CD4$^+$ T-cells.	0.5 day^{-1}
ϵ_V	Rate of generation for HIV virions by infected CD4$^+$ T-cells.	100 vir. cell^{-1} day^{-1}
μ_V	Rate of the death for infectious virus.	3 day^{-1}
α	Rate of eliminating the infected cells by the activated CD8$^+$ T-cells.	0.02 day^{-1}
λ_Z	Rate of recruiting the CD8$^+$ T-cells per unit time.	20 cell/mm^3/day
μ_Z	Rate of the death for CD8$^+$ T-cells.	0.06 day^{-1}
β	Rate of activation for CD8$^+$ T-cells due to the attendance the infected CD4$^+$ T-cells.	0.004 day^{-1}
μ_{Z_a}	Rate of decaying for activated defence cells decay per unit time.	0.004 day^{-1}

2. Homotopy analysis transform method

Defining the linear operators $L_T, L_I, L_V, L_Z, L_{Z_a}$ as follows

$$L_T = L_I = L_V = L_Z = L_{Z_a} = \mathcal{L},$$

where $\mathcal{L}$ is the Laplace transformation. Applying this operator for non-linear system of Eqs. (1) as

$$\begin{aligned}
\mathcal{L}[T(t)] &= \frac{T(0)}{s} + \frac{\mathcal{L}[\lambda_T]}{s} - \frac{\mu_T}{s}\mathcal{L}[T(t)] - \frac{\chi}{s}\mathcal{L}[T(t)V(t)],\\
\mathcal{L}[I(t)] &= \frac{I(0)}{s} + \frac{\chi}{s}\mathcal{L}[T(t)V(t)] - \frac{\mu_I}{s}\mathcal{L}[I(t)] - \frac{\alpha}{s}\mathcal{L}[I(t)Z_a(t)],\\
\mathcal{L}[V(t)] &= \frac{V(0)}{s} + \frac{\epsilon_V\mu_I}{s}\mathcal{L}[I(t)] - \frac{\mu_V}{s}\mathcal{L}[V(t)],\\
\mathcal{L}[Z(t)] &= \frac{Z(0)}{s} + \frac{\mathcal{L}[\lambda_Z]}{s} + \frac{\mu_Z}{s}\mathcal{L}[Z(t)] - \frac{\beta}{s}\mathcal{L}[Z(t)I(t)],\\
\mathcal{L}[Z_a(t)] &= \frac{Z_a(0)}{s} + \frac{\beta}{s}\mathcal{L}[Z(t)I(t)] - \frac{\mu_{Z_a}}{s}\mathcal{L}[Z_a(t)].
\end{aligned}$$

Let $0 \leqslant q \leqslant 1$ be an embedding parameter, $\hbar$ is an auxiliary parameter, $H_T(t), H_I(t), H_V(t), H_Z(t)$ and $H_{Z_a}(t)$ are the auxiliary functions, $L_T, L_I, L_V, L_Z, L_{Z_a}$ are the linear operators and $N_T, N_I, N_V, N_Z, N_{Z_a}$ are the non-linear operators then the following Homotopy maps can be defined as

$$\begin{aligned}
&H_T[\hat{T}(t;q), \hat{I}(t;q), \hat{V}(t;q), \hat{Z}(t;q), \hat{Z}_a(t;q)]\\
&\quad = (1-q)L_T[\hat{T}(t;q) - T_0(t)] - q\hbar H_T(t)N_T[\hat{T}(t;q), \hat{I}(t;q), \hat{V}(t;q), \hat{Z}(t;q), \hat{Z}_a(t;q)],\\
&H_I[\hat{T}(t;q), \hat{I}(t;q), \hat{V}(t;q), \hat{Z}(t;q), \hat{Z}_a(t;q)]\\
&\quad = (1-q)L_I[\hat{I}(t;q) - I_0(t)] - q\hbar H_I(t)N_I[\hat{T}(t;q), \hat{I}(t;q), \hat{V}(t;q), \hat{Z}(t;q), \hat{Z}_a(t;q)],\\
&H_V[\hat{T}(t;q), \hat{I}(t;q), \hat{V}(t;q), \hat{Z}(t;q), \hat{Z}_a(t;q)]\\
&\quad = (1-q)L_V[\hat{V}(t;q) - V_0(t)] - q\hbar H_V(t)N_V[\hat{T}(t;q), \hat{I}(t;q), \hat{V}(t;q), \hat{Z}(t;q), \hat{Z}_a(t;q)],\\
&H_Z[\hat{T}(t;q), \hat{I}(t;q), \hat{V}(t;q), \hat{Z}(t;q), \hat{Z}_a(t;q)]\\
&\quad = (1-q)L_Z[\hat{Z}(t;q) - Z_0(t)] - q\hbar H_Z(t)N_Z[\hat{T}(t;q), \hat{I}(t;q), \hat{V}(t;q), \hat{Z}(t;q), \hat{Z}_a(t;q)],\\
&H_{Z_a}[\hat{T}(t;q), \hat{I}(t;q), \hat{V}(t;q), \hat{Z}(t;q), \hat{Z}_a(t;q)]\\
&\quad = (1-q)L_{Z_a}[\hat{Z}_a(t;q) - Z_{a0}(t)] - q\hbar H_{Z_a}(t)N_{Z_a}[\hat{T}(t;q), \hat{I}(t;q), \hat{V}(t;q), \hat{Z}(t;q), \hat{Z}_a(t;q)],
\end{aligned}$$

where the non-linear operators $N_T, N_I, N_V, N_Z, N_{Z_a}$ are defined in the following forms

$$N_T[\hat{T}(t;q),\hat{I}(t;q),\hat{V}(t;q),\hat{Z}(t;q),\hat{Z}_a(t;q)]=\frac{\partial\hat{T}(t;q)}{\partial t}-\lambda_T+\mu_T\hat{T}(t;q)+\chi\hat{T}(t;q)\hat{V}(t;q),$$
$$N_I[\hat{T}(t;q),\hat{I}(t;q),\hat{V}(t;q),\hat{Z}(t;q),\hat{Z}_a(t;q)]=\frac{\partial\hat{I}(t;q)}{\partial t}-\chi\hat{T}(t;q)\hat{V}(t;q)+\mu_I\hat{I}(t;q)+\alpha\hat{I}(t;q)\hat{Z}_a(t;q),$$
$$N_V[\hat{T}(t;q),\hat{I}(t;q),\hat{V}(t;q),\hat{Z}(t;q),\hat{Z}_a(t;q)]=\frac{\partial\hat{V}(t;q)}{\partial t}-\epsilon_V\mu_I\hat{I}(t;q)+\mu_V\hat{V}(t;q),$$
$$N_Z[\hat{T}(t;q),\hat{I}(t;q),\hat{V}(t;q),\hat{Z}(t;q),\hat{Z}_a(t;q)]=\frac{\partial\hat{Z}(t;q)}{\partial t}-\lambda_Z+\mu_Z\hat{Z}(t;q)+\beta\hat{Z}(t;q)\hat{I}(t;q),$$
$$N_{Z_a}[\hat{T}(t;q),\hat{I}(t;q),\hat{V}(t;q),\hat{Z}(t;q),\hat{Z}_a(t;q)]=\frac{\partial\hat{Z}_a(t;q)}{\partial t}-\beta\hat{Z}(t;q)\hat{I}(t;q)+\mu_{Z_a}\hat{Z}_a(t;q).$$

Now, we can construct the following zero order deformation equations as

$$(1-q)L_T[\hat{T}(t;q)-T_0(t)]-q\hbar H_T(t)N_T[\hat{T}(t;q),\hat{I}(t;q),\hat{V}(t;q),\hat{Z}(t;q),\hat{Z}_a(t;q)]=0,$$
$$(1-q)L_I[\hat{I}(t;q)-I_0(t)]-q\hbar H_I(t)N_I[\hat{T}(t;q),\hat{I}(t;q),\hat{V}(t;q),\hat{Z}(t;q),\hat{Z}_a(t;q)]=0,$$
$$(1-q)L_V[\hat{V}(t;q)-V_0(t)]-q\hbar H_V(t)N_V[\hat{T}(t;q),\hat{I}(t;q),\hat{V}(t;q),\hat{Z}(t;q),\hat{Z}_a(t;q)]=0,$$
$$(1-q)L_Z[\hat{Z}(t;q)-Z_0(t)]-q\hbar H_Z(t)N_Z[\hat{T}(t;q),\hat{I}(t;q),\hat{V}(t;q),\hat{Z}(t;q),\hat{Z}_a(t;q)]=0,$$
$$(1-q)L_{Z_a}[\hat{Z}_a(t;q)-Z_{a0}(t)]-q\hbar H_{Z_a}(t)N_{Z_a}[\hat{T}(t;q),\hat{I}(t;q),\hat{V}(t;q),\hat{Z}(t;q),\hat{Z}_a(t;q)]=0.$$

Using the following Taylor expansions as

$$\hat{T}(t;q)=T_0(t)+\sum_{m=1}^{\infty}T_m(t)q^m,\qquad \hat{I}(t;q)=I_0(t)+\sum_{m=1}^{\infty}I_m(t)q^m,$$
$$\hat{V}(t;q)=V_0(t)+\sum_{m=1}^{\infty}V_m(t)q^m,\qquad \hat{Z}(t;q)=Z_0(t)+\sum_{m=1}^{\infty}Z_m(t)q^m,$$
$$\hat{Z}_a(t;q)=Z_{a0}(t)+\sum_{m=1}^{\infty}Z_{am}(t)q^m,$$

where

$$T_m=\frac{1}{m!}\frac{\partial^m\hat{T}(t;q)}{\partial q^m}\bigg|_{q=0},\ I_m=\frac{1}{m!}\frac{\partial^m\hat{I}(t;q)}{\partial q^m}\bigg|_{q=0},\qquad V_m=\frac{1}{m!}\frac{\partial^m\hat{V}(t;q)}{\partial q^m}\bigg|_{q=0},$$

$$Z_m=\frac{1}{m!}\frac{\partial^m\hat{Z}(t;q)}{\partial q^m}\bigg|_{q=0},\ Z_{am}=\frac{1}{m!}\frac{\partial^m\hat{Z}_a(t;q)}{\partial q^m}\bigg|_{q=0}.$$

Defining the following vectors

$$\hat{T}_m(t)=\Big\{T_0(t),T_1(t),\ldots,T_m(t)\Big\},\qquad \hat{I}_m(t)=\Big\{I_0(t),I_1(t),\ldots,I_m(t)\Big\},$$
$$\hat{V}_m(t)=\Big\{V_0(t),V_1(t),\ldots,V_m(t)\Big\},\qquad \hat{Z}_m(t)=\Big\{Z_0(t),Z_1(t),\ldots,Z_m(t)\Big\},$$
$$\hat{Z}_{am}(t)=\Big\{Z_{a0}(t),Z_{a1}(t),\ldots,Z_{am}(t)\Big\},$$

to construct the m-th order deformation equations as follows

$$
\begin{aligned}
&L_T\left[T_m(t)-\chi_m T_{m-1}(t)\right]=\hbar H_T(t)\Re_m^T\left(\vec{T}_{m-1},\vec{I}_{m-1},\vec{V}_{m-1},\vec{Z}_{m-1},\vec{Z}_{am-1}\right),\\
&L_I\left[I_m(t)-\chi_m I_{m-1}(t)\right]=\hbar H_I(t)\Re_m^I\left(\vec{T}_{m-1},\vec{I}_{m-1},\vec{V}_{m-1},\vec{Z}_{m-1},\vec{Z}_{am-1}\right),\\
&L_V\left[V_m(t)-\chi_m V_{m-1}(t)\right]=\hbar H_V(t)\Re_m^V\left(\vec{T}_{m-1},\vec{I}_{m-1},\vec{V}_{m-1},\vec{Z}_{m-1},\vec{Z}_{am-1}\right),\\
&L_Z\left[Z_m(t)-\chi_m Z_{m-1}(t)\right]=\hbar H_Z(t)\Re_m^Z\left(\vec{T}_{m-1},\vec{I}_{m-1},\vec{V}_{m-1},\vec{Z}_{m-1},\vec{Z}_{am-1}\right),\\
&L_{Z_a}\left[Z_{am}(t)-\chi_m Z_{am-1}(t)\right]=\hbar H_{Z_a}(t)\Re_m^{Z_a}\left(\vec{T}_{m-1},\vec{I}_{m-1},\vec{V}_{m-1},\vec{Z}_{m-1},\vec{Z}_{am-1}\right),
\end{aligned}
\tag{2}
$$

where

$$
\begin{aligned}
&\Re_m^T=\mathcal{L}[T_{m-1}]-\frac{T_{m-1(0)}}{s}-(1-\chi_m)\frac{\mathcal{L}[\lambda_T]}{s}+\frac{\mu_T}{s}\mathcal{L}[T_{m-1}(t)]+\frac{\chi}{s}\mathcal{L}\left[\sum_{j=0}^{m-1}T_j(t)V_{m-1-j}(t)\right],\\
&\Re_m^I=\mathcal{L}[I_{m-1}]-\frac{I_{m-1(0)}}{s}-\frac{\chi}{s}\mathcal{L}\left[\sum_{j=0}^{m-1}T_j(t)V_{m-1-j}(t)\right]+\frac{\mu_I}{s}\mathcal{L}[I_{m-1}(t)]+\frac{\alpha}{s}\mathcal{L}\left[\sum_{j=0}^{m-1}I_j(t)Z_{am-1-j}(t)\right],\\
&\Re_m^V=\mathcal{L}[V_{m-1}]-\frac{V_{m-1(0)}}{s}-\frac{\epsilon_V\mu_I}{s}\mathcal{L}[I_{m-1}(t)]+\frac{\mu_V}{s}\mathcal{L}[V_{m-1}(t)],\\
&\Re_m^Z=\mathcal{L}[Z_{m-1}]-\frac{Z_{m-1(0)}}{s}-(1-\chi_m)\frac{\mathcal{L}[\lambda_Z]}{s}+\frac{\mu_Z}{s}\mathcal{L}[Z_{m-1}(t)]+\frac{\beta}{s}\mathcal{L}\left[\sum_{j=0}^{m-1}Z_j(t)I_{m-1-j}(t)\right],\\
&\Re_m^{Z_a}=\mathcal{L}[Z_{am-1}]-\frac{Z_{am-1(0)}}{s}-\frac{\beta}{s}\mathcal{L}\left[\sum_{j=0}^{m-1}Z_j(t)I_{m-1-j}(t)\right]+\frac{\mu_{Z_a}}{s}\mathcal{L}[Z_{am-1}(t)],
\end{aligned}
$$

and

$$
\chi_m=\begin{cases}0, & m\leqslant 1\\[2ex] 1, & m>1.\end{cases}
$$

Applying the inverse Laplace transformation $\mathcal{L}^{-1}$ for Eqs. (2) we get

$$
\begin{aligned}
&T_m(t)=\chi_m T_{m-1}(t)+\hbar\mathcal{L}^{-1}\left[\Re_m^T(t)\right], \qquad I_m(t)=\chi_m I_{m-1}(t)+\hbar\mathcal{L}^{-1}\left[\Re_m^I(t)\right],\\
&V_m(t)=\chi_m V_{m-1}(t)+\hbar\mathcal{L}^{-1}\left[\Re_m^V(t)\right], \qquad Z_m(t)=\chi_m Z_{m-1}(t)+\hbar\mathcal{L}^{-1}\left[\Re_m^Z(t)\right],\\
&Z_{am}(t)=\chi_m Z_{am-1}(t)+\hbar\mathcal{L}^{-1}\left[\Re_m^{Z_a}(t)\right],
\end{aligned}
$$

and finally the approximate solutions can be obtained by

$$
\begin{aligned}
&T_m(t)=\sum_{j=0}^{m}T_j(t), \qquad I_m(t)=\sum_{j=0}^{m}I_j(t), \qquad V_m(t)=\sum_{j=0}^{m}V_j(t),\\
&Z_m(t)=\sum_{j=0}^{m}Z_j(t), \qquad Z_{am}(t)=\sum_{j=0}^{m}Z_{aj}(t).
\end{aligned} \tag{3}
$$

3. Convergence theorem

By proving the following theorem, we can show the capabilities of the HATM to solve the non-linear system of Eqs. (1).

Theorem 3.1 *Let series solutions (3) be convergent that are constructed by the m-th order deformation Eqs. (2). They must be the exact solution of system (1).*

Proof Let the series solutions (3) be convergent. Hence, if

$$
\begin{aligned}
&P_1(t)=\sum_{m=0}^{\infty}T_m(t),\ P_2(t)=\sum_{m=0}^{\infty}I_m(t),\quad P_3(t)=\sum_{m=0}^{\infty}V_m(t),\\
&P_4(t)=\sum_{m=0}^{\infty}Z_m(t),\ P_5(t)=\sum_{m=0}^{\infty}Z_{am}(t),
\end{aligned}
$$

then

$$
\begin{aligned}
&\lim_{m\to\infty}T_m(t)=0,\ \ \lim_{m\to\infty}I_m(t)=0,\\
&\lim_{m\to\infty}V_m(t)=0,\ \ \lim_{m\to\infty}Z_m(t)=0,\\
&\lim_{m\to\infty}Z_{am}(t)=0.
\end{aligned} \tag{4}
$$

So, we can write

$$
\begin{aligned}
&\sum_{m=1}^{N}\Big[T_m(t)-\chi_m T_{m-1}(t)\Big]=T_N(t), \qquad \sum_{m=1}^{N}\Big[I_m(t)-\chi_m I_{m-1}(t)\Big]=I_N(t),\\
&\sum_{m=1}^{N}\Big[V_m(t)-\chi_m v_{m-1}(t)\Big]=V_N(t), \qquad \sum_{m=1}^{N}\Big[Z_m(t)-\chi_m Z_{m-1}(t)\Big]=Z_N(t),\\
&\sum_{m=1}^{N}\Big[Z_{am}(t)-\chi_m Z_{am-1}(t)\Big]=Z_{aN}(t),
\end{aligned} \tag{5}
$$

where Eqs. (4) and (5) are applied to construct the following relations as follows

$$
\begin{aligned}
&\sum_{m=1}^{N}\Big[T_m(t)-\chi_m T_{m-1}(t)\Big]=\lim_{N\to\infty}T_N(t)=0,\\
&\sum_{m=1}^{N}\Big[I_m(t)-\chi_m I_{m-1}(t)\Big]=\lim_{N\to\infty}I_N(t)=0,\\
&\sum_{m=1}^{N}\Big[V_m(t)-\chi_m V_{m-1}(t)\Big]=\lim_{N\to\infty}V_N(t)=0,\\
&\sum_{m=1}^{N}\Big[Z_m(t)-\chi_m Z_{m-1}(t)\Big]=\lim_{N\to\infty}Z_N(t)=0,\\
&\sum_{m=1}^{N}\Big[Z_{am}(t)-\chi_m Z_{am-1}(t)\Big]=\lim_{N\to\infty}Z_{aN}(t)=0.
\end{aligned}
$$

Applying the linear operators L_T, L_I, L_V, L_Z and L_{Z_a} as

$$
\begin{aligned}
&\sum_{m=1}^{\infty}L_T\Big[T_m(t)-\chi_m T_{m-1}(t)\Big]=L_T\Big[\sum_{m=1}^{\infty}T_m(t)-\chi_m T_{m-1}(t)\Big]=0,\\
&\sum_{m=1}^{\infty}L_I\Big[I_m(t)-\chi_m I_{m-1}(t)\Big]=L_I\Big[\sum_{m=1}^{\infty}I_m(t)-\chi_m I_{m-1}(t)\Big]=0,\\
&\sum_{m=1}^{\infty}L_V\Big[V_m(t)-\chi_m V_{m-1}(t)\Big]=L_V\Big[\sum_{m=1}^{\infty}V_m(t)-\chi_m V_{m-1}(t)\Big]=0,\\
&\sum_{m=1}^{\infty}L_Z\Big[Z_m(t)-\chi_m Z_{m-1}(t)\Big]=L_Z\Big[\sum_{m=1}^{\infty}Z_m(t)-\chi_m Z_{m-1}(t)\Big]=0,\\
&\sum_{m=1}^{\infty}L_{Z_a}\Big[Z_{am}(t)-\chi_m Z_{am-1}(t)\Big]=L_{Z_a}\Big[\sum_{m=1}^{\infty}Z_{am}(t)-\chi_m Z_{am-1}(t)\Big]=0.
\end{aligned}
\tag{6}
$$

By using Eqs. (2) and (6) we get

$$
\begin{aligned}
&\hbar H_T(t)\sum_{m=1}^{\infty}\Re_m^{T}(\vec{T}_{m-1},\vec{I}_{m-1},\vec{V}_{m-1},\vec{Z}_{m-1},\vec{Z}_{am-1})=0,\\
&\hbar H_I(t)\sum_{m=1}^{\infty}\Re_m^{I}(\vec{T}_{m-1},\vec{I}_{m-1},\vec{V}_{m-1},\vec{Z}_{m-1},\vec{Z}_{am-1})=0,\\
&\hbar H_V(t)\sum_{m=1}^{\infty}\Re_m^{V}(\vec{T}_{m-1},\vec{I}_{m-1},\vec{V}_{m-1},\vec{Z}_{m-1},\vec{Z}_{am-1})=0,\\
&\hbar H_Z(t)\sum_{m=1}^{\infty}\Re_m^{Z}(\vec{T}_{m-1},\vec{I}_{m-1},\vec{V}_{m-1},\vec{Z}_{m-1},\vec{Z}_{am-1})=0,\\
&\hbar H_{Z_a}(t)\sum_{m=1}^{\infty}\Re_m^{Z_a}(\vec{T}_{m-1},\vec{I}_{m-1},\vec{V}_{m-1},\vec{Z}_{m-1},\vec{Z}_{am-1})=0.
\end{aligned}
\tag{7}
$$

According to the base definitions of the HAM in Eqs. (7), $\hbar \neq 0, H_S(t) \neq$

$0, H_I(t) \neq 0, H_R(t) \neq 0$, thus

$$\begin{aligned}
&\sum_{m=1}^{\infty} \Re_m^T(\vec{T}_{m-1}, \vec{I}_{m-1}, \vec{V}_{m-1}, \vec{Z}_{m-1}, \vec{Z}_{am-1}) = 0,\\
&\sum_{m=1}^{\infty} \Re_m^I(\vec{T}_{m-1}, \vec{I}_{m-1}, \vec{V}_{m-1}, \vec{Z}_{m-1}, \vec{Z}_{am-1}) = 0,\\
&\sum_{m=1}^{\infty} \Re_m^V(\vec{T}_{m-1}, \vec{I}_{m-1}, \vec{V}_{m-1}, \vec{Z}_{m-1}, \vec{Z}_{am-1}) = 0,\\
&\sum_{m=1}^{\infty} \Re_m^Z(\vec{T}_{m-1}, \vec{I}_{m-1}, \vec{V}_{m-1}, \vec{Z}_{m-1}, \vec{Z}_{am-1}) = 0,\\
&\sum_{m=1}^{\infty} \Re_m^{Z_a}(\vec{T}_{m-1}, \vec{I}_{m-1}, \vec{V}_{m-1}, \vec{Z}_{m-1}, \vec{Z}_{am-1}) = 0.
\end{aligned} \tag{8}$$

Substituting $\Re_m^T, \Re_m^I, \Re_m^V, \Re_m^Z$ and $\Re_m^{Z_a}$ into Eqs. (8) and assuming $(.)' = \frac{d}{dt}$ the following formulas are obtained as

$$\begin{aligned}
&\sum_{m=1}^{\infty} \Re_m^T = \sum_{m=1}^{\infty} \left[T'_{m-1} - (1-\chi_m)\lambda_T + \mu_T T_{m-1}(t) + \chi \sum_{j=0}^{m-1} T_j(t) V_{m-1-j}(t) \right]\\
&= \sum_{m=0}^{\infty} T'_m - \lambda_T + \mu_T \sum_{m=0}^{\infty} T_m(t) + \chi \sum_{m=1}^{\infty} \sum_{j=0}^{m-1} T_j(t) V_{m-1-j}(t)\\
&= \sum_{m=0}^{\infty} T'_m - \lambda_T + \mu_T \sum_{m=0}^{\infty} T_m(t) + \chi \sum_{j=0}^{\infty} \sum_{m=j+1}^{\infty} T_j(t) V_{m-1-j}(t)\\
&= \sum_{m=0}^{\infty} T'_m - \lambda_T + \mu_T \sum_{m=0}^{\infty} T_m(t) + \chi \sum_{j=0}^{\infty} T_j(t) \sum_{m=0}^{\infty} V_m(t)\\
&= P'_1(t) - \lambda_T + \mu_T P_1(t) + \chi P_1(t) P_3(t),
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
&\sum_{m=1}^{\infty} \Re_m^I = \sum_{m=1}^{\infty} \left[I'_{m-1} - \chi \sum_{j=0}^{m-1} T_j(t) V_{m-1-j}(t) + \mu_I I_{m-1}(t) + \alpha \sum_{j=0}^{m-1} I_j(t) Z_{am-1-j}(t) \right]\\
&= \sum_{m=0}^{\infty} I'_m - \chi \sum_{m=1}^{\infty} \sum_{j=0}^{m-1} T_j(t) V_{m-1-j}(t) + \mu_I \sum_{m=0}^{\infty} I_m(t) + \alpha \sum_{m=1}^{\infty} \sum_{j=0}^{m-1} I_j(t) Z_{am-1-j}(t)\\
&= \sum_{m=0}^{\infty} I'_m - \chi \sum_{j=0}^{\infty} \sum_{m=j+1}^{\infty} T_j(t) V_{m-1-j}(t) + \mu_I \sum_{m=0}^{\infty} I_m(t) + \alpha \sum_{j=0}^{\infty} \sum_{m=j+1}^{\infty} I_j(t) Z_{am-1-j}(t)\\
&= \sum_{m=0}^{\infty} I'_m - \chi \sum_{j=0}^{\infty} T_j(t) \sum_{m=0}^{\infty} V_m(t) + \mu_I \sum_{m=0}^{\infty} I_m(t) + \alpha \sum_{j=0}^{\infty} I_j(t) \sum_{m=0}^{\infty} Z_{am}(t)\\
&= P'_2(t) - \chi P_1(t) P_3(t) + \mu_I P_2(t) + \alpha P_2(t) P_5(t),
\end{aligned} \tag{10}$$

and

$$\begin{aligned}\sum_{m=1}^{\infty}\Re_m^V &= \sum_{m=1}^{\infty}\left[V'_{m-1} - \epsilon_V\mu_I I_{m-1}(t) + \mu_V V_{m-1}(t)\right]\\ &= \sum_{m=0}^{\infty}V'_m - \epsilon_V\mu_I\sum_{m=0}^{\infty}I_m(t) + \mu_V\sum_{m=0}^{\infty}V_m(t)\\ &= P'_3(t) - \epsilon_V\mu_I P_2(t) + \mu_V P_3(t),\end{aligned} \tag{11}$$

and

$$\begin{aligned}\sum_{m=1}^{\infty}\Re_m^Z &= \sum_{m=1}^{\infty}\left[Z'_{m-1} - (1-\chi_m)\lambda_Z + \mu_Z Z_{m-1}(t) + \beta\sum_{j=0}^{m-1}Z_j(t)I_{m-1-j}(t)\right]\\ &= \sum_{m=0}^{\infty}Z'_m - \lambda_Z + \mu_Z\sum_{m=0}^{\infty}Z_m(t) + \beta\sum_{j=0}^{\infty}\sum_{m=j+1}^{\infty}Z_j(t)I_{m-1-j}(t)\\ &= \sum_{m=0}^{\infty}Z'_m - \lambda_Z + \mu_Z\sum_{m=0}^{\infty}Z_m(t) + \beta\sum_{j=0}^{\infty}Z_j(t)\sum_{m=0}^{\infty}I_m(t)\\ &= P'_4(t) - \lambda_Z + \mu_Z P_4(t) + \beta P_4(t)P_2(t),\end{aligned} \tag{12}$$

and

$$\begin{aligned}\sum_{m=1}^{\infty}\Re_m^{Z_a} &= \sum_{m=1}^{\infty}\left[{Z_a}'_{m-1} - \beta\sum_{j=0}^{m-1}Z_j(t)I_{m-1-j}(t) + \mu_{Z_a}Z_{a\,m-1}(t)\right]\\ &= \sum_{m=0}^{\infty}{Z_a}'_m - \beta\sum_{m=1}^{\infty}\sum_{j=0}^{m-1}Z_j(t)I_{m-1-j}(t) + \mu_{Z_a}\sum_{m=0}^{\infty}Z_{am}(t)\\ &= \sum_{m=0}^{\infty}{Z_a}'_m - \beta\sum_{j=0}^{\infty}\sum_{m=j+1}^{\infty}Z_j(t)I_{m-1-j}(t) + \mu_{Z_a}\sum_{m=0}^{\infty}Z_{am}(t)\\ &= \sum_{m=0}^{\infty}{Z_a}'_m - \beta\sum_{j=0}^{\infty}Z_j(t)\sum_{m=0}^{\infty}I_m(t) + \mu_{Z_a}\sum_{m=0}^{\infty}Z_{am}(t)\\ &= P'_5(t) - \beta P_4(t)P_2(t) + \mu_{Z_a}P_5(t).\end{aligned} \tag{13}$$

Eqs. (9), (10), (11), (12) and (13) show that the series solutions $P_1(t), P_2(t), P_3(t), P_4(t)$ and $P_5(t)$ must be the exact solutions of Eqs. (1). ■

4. Numerical illustration

In this section, in order to show the flexibility of HATM to solve the non-linear bio-mathematical model (1), the numerical solutions for $N = 5$ are presented as follows

$$\begin{aligned}
T_5(t) =&1000 + 0.12\hbar t + 0.24\hbar^2 t + 0.24\hbar^3 t + 0.12\hbar^4 t + 0.024\hbar^5 t + 0.361203\hbar^2 t^2\\
&+ 0.722406\hbar^3 t^2 + 0.541804\hbar^4 t^2 + 0.144481\hbar^5 t^2 + 0.409213\hbar^3 t^3 + 0.613819\hbar^4 t^3\\
&+ 0.245528\hbar^5 t^3 + 0.17452\hbar^4 t^4 + 0.139616\hbar^5 t^4 + 0.0238202\hbar^5 t^5,\\
I_5(t) =& - 0.12\hbar t - 0.24\hbar^2 t - 0.24\hbar^3 t - 0.12\hbar^4 t - 0.024\hbar^5 t - 0.420003\hbar^2 t^2\\
&- 0.840006\hbar^3 t^2 - 0.630004\hbar^4 t^2 - 0.168001\hbar^5 t^2 - 0.478009\hbar^3 t^3 - 0.717014\hbar^4 t^3\\
&- 0.286805\hbar^5 t^3 - 0.203898\hbar^4 t^4 - 0.163119\hbar^5 t^4 - 0.0278351\hbar^5 t^5,\\
V_5(t) =&1 + 15\hbar t + 30.\hbar^2 t + 30.\hbar^3 t + 15.\hbar^4 t + 3.\hbar^5 t + 51.\hbar^2 t^2 + 102.\hbar^3 t^2\\
&+ 76.5\hbar^4 t^2 + 20.4\hbar^5 t^2 + 58.\hbar^3 t^3 + 87.0001\hbar^4 t^3 + 34.8\hbar^5 t^3\\
&+ 24.7376\hbar^4 t^4 + 19.7901\hbar^5 t^4 + 3.37631\hbar^5 t^5,\\
Z_5(t) =&500 + 50.\hbar t + 100.\hbar^2 t + 100.\hbar^3 t + 50.\hbar^4 t + 10.\hbar^5 t + 2.76\hbar^2 t^2 + 5.52\hbar^3 t^2\\
&+ 4.14\hbar^4 t^2 + 1.104\hbar^5 t^2 - 0.228002\hbar^3 t^3 - 0.342003\hbar^4 t^3 - 0.136801\hbar^5 t^3\\
&- 0.123345\hbar^4 t^4 - 0.0986763\hbar^5 t^4 - 0.0169991\hbar^5 t^5,\\
Z_{a5}(t) =&0.24\hbar^2 t^2 + 0.48\hbar^3 t^2 + 0.36\hbar^4 t^2 + 0.096\hbar^5 t^2 + 0.283522\hbar^3 t^3 + 0.425283\hbar^4 t^3\\
&+ 0.170113\hbar^5 t^3 + 0.121777\hbar^4 t^4 + 0.0974217\hbar^5 t^4 + 0.0167226\hbar^5 t^5,
\end{aligned}$$

The regions of convergence are shown by several $\hbar$-curves for $N = 5, 10$ and $t = 1$ in Figures 3 and 4. These regions are parallel parts of $\hbar$-curves with axiom x. So for $N = 5$ and $t = 1$ the convergence regions are

$$\begin{gathered}
-0.9 \leqslant \hbar_T \leqslant -0.2,\\
-0.8 \leqslant \hbar_I \leqslant -0.2,\\
-0.8 \leqslant \hbar_V \leqslant -0.2,\\
-1.2 \leqslant \hbar_Z \leqslant -0.6,\\
-0.8 \leqslant \hbar_{Z_a} \leqslant -0.4,
\end{gathered}$$

and for $N = 10$ we get

$$\begin{gathered}
-0.9 \leqslant \hbar_T \leqslant -0.4,\\
-1 \leqslant \hbar_I \leqslant -0.2,\\
-1 \leqslant \hbar_V \leqslant -0.3,\\
-1.2 \leqslant \hbar_Z \leqslant -0.4,\\
-1 \leqslant \hbar_{Z_a} \leqslant -0.4.
\end{gathered}$$

Also, the following residual error functions are applied to show the accuracy of presented method as

$$
\begin{aligned}
E_{N,T}(t) &= \frac{dT(t)}{dt} - \lambda_T + \mu_T T(t) + \chi T(t)V(t)\\
E_{N,I}(t) &= \frac{dI(t)}{dt} - \chi T(t)V(t) + \mu_I I(t) + \alpha I(t) Z_a(t),\\
E_{N,V}(t) &= \frac{dV(t)}{dt} - \epsilon_V \mu_I I(t) + \mu_V V(t),\\
E_{N,Z}(t) &= \frac{dZ(t)}{dt} - \lambda_Z + \mu_Z Z(t) + \beta Z(t) I(t),\\
E_{N,Z_a}(t) &= \frac{dZ_a(t)}{dt} - \beta Z(t) I(t) + \mu_{Z_a} Z_a(t),
\end{aligned}
$$

and the plots of error functions are demonstrated in Figure 5 for $N = 5, 10$ and $\hbar = -0.8$.

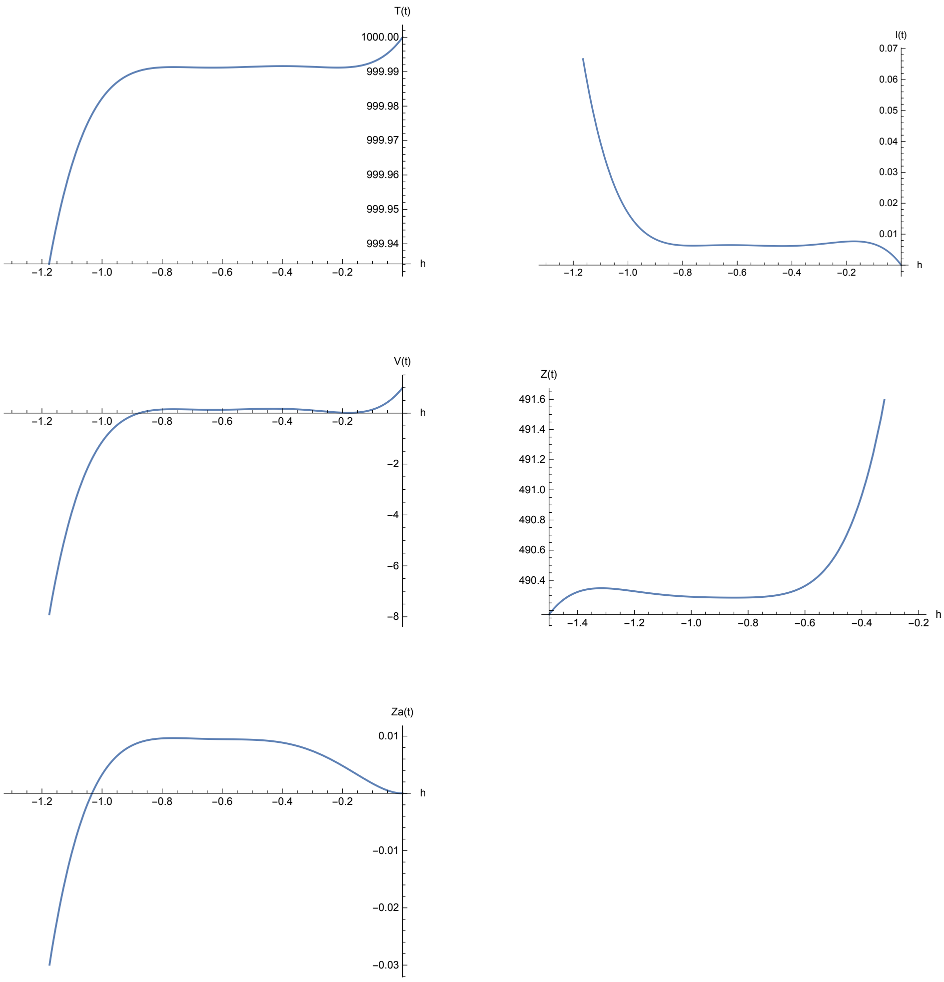

Figure 3. $\hbar$-curves of $T(t), I(t), V(t), Z(t)$ and $Z_a(t)$ for $N = 5, t = 1$.

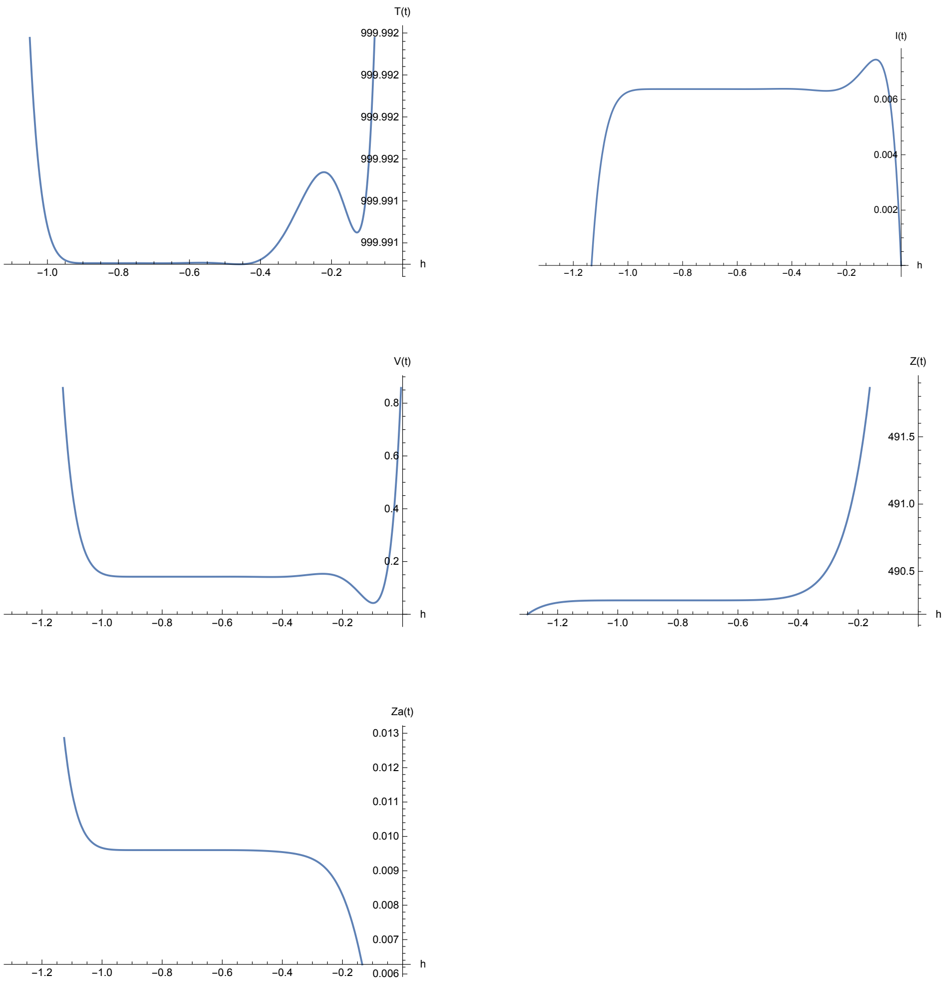

Figure 4. $\hbar$-curves of $T(t), I(t), V(t), Z(t)$ and $Z_a(t)$ for $N = 10, t = 1$.

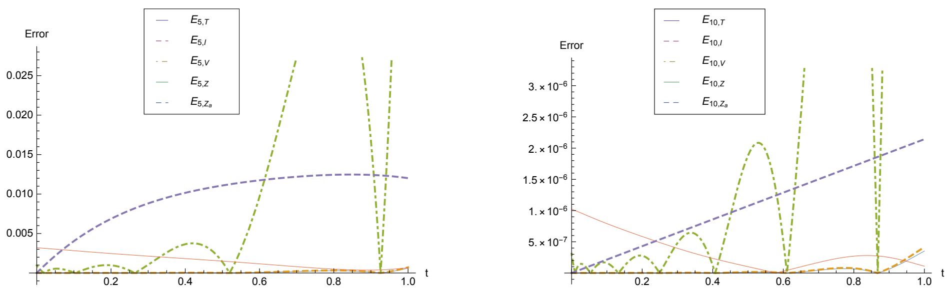

Figure 5. Residual error functions for $N = 5, 10$ and $\hbar = 0.8$.

5. Conclusion

The HATM is among of the accurate semi-analytical methods for solving linear and non-linear problems based on its capabilities such as operators, functions and parameters that we have freedom to chose them. In this research, the HATM was applied to solve the bio-mathematical model of HIV infection for CD8$^+$ T-cells. Furthermore, the convergence theorem was proved that shows the competency of HATM for solving non-linear problems. Based on the numerical solutions for $N = 5, 10$ several $\hbar$-curves were plotted that show the convergence regions of solutions. The precision of method were demonstrated by plotting the residual error functions.

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