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Inequalities of Hermite-Hadamard-Mercer Type for Convex Functions via K-Fractional Integrals

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Abstract. In this paper, the authors establish inequalities of the Hermite-Hadamard-Mercer type for convex functions by applying k-fractional integrals. We prove some new fractional inequalities connected to the left part of Hermite-Hadamard-Mercer type inequalities for differentiable mappings whose first derivatives in absolute value are convex.

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1. Introduction

The situation of the fractional calculus (integrals and derivatives) has won vast popularity and significance throughout the previous there decades or so, due generally to its demonstrated applications in numerous seemingly numerous and great fields of science and engineering. Let $0 < x_1 \leqslant x_2 \leqslant ... \leqslant x_n$ and let $\mu_i = (\mu_1, \mu_2, ...\mu_n)$ nonnegative weights such that $\Sigma_{i=1}^{n} \mu_i = 1$. The Jensen inequality [7] states that f

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is a convex function on the interval $[a,b]$; then,

$$f\left(\Sigma_{i=1}^{n}\mu_i x_i\right) \leqslant \Sigma_{i=1}^{n}\mu_i f(x_i),$$

where for all $x_i \in [a,b]$ and $\mu_i \in [0,1]$, $(i=\overline{1,n})$. The Hermite–Hadamard inequality states that if a mapping $f: I \subseteq R \to R$ is a convex function on I with $a,b \in I$, $a<b$ then,

$$f\Big(\frac{a+b}{2}\Big) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1}$$

The double inequality holds in the reversed direction if f is concave (see, [6]).

Theorem 1.1 *[11] If f is convex function on $I=[a,b]$, then*

$$f\left(a+b-\Sigma_{i=1}^{n}\mu_i x_i\right) \leqslant f(a)+f(b)-\Sigma_{i=1}^{n}\mu_i f(x_i),$$

for each $x_i \in [a,b]$ and $\mu_i \in [0,1]$, $(i=\overline{1,n})$ with $\Sigma_{i=1}^{n}\mu_i = 1$. For some results related with Jensen-Mercer inequality, (see, [3, 9–11, 14]).

After these necessary inequalities about convex functions, we will now give the definitions which we will use in this paper.

Definition 1.2 [4, 17, 19, 20] Let $f \in L[a,b]$. The Left sided and right sided Riemann-Liouville fractional integrals $J_{a^+}^{\alpha}f$ and $J_{b^-}^{\alpha}f$ of order $\alpha>0$ with $a \geqslant 0$ can be defined respectively by

$$J_{a^+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \ x>a$$

$$J_{b^-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad x<b$$

here $\Gamma(\alpha)$ is Euler Gamma function. and $J_{a^+}^{0}f(x) = J_{b^-}^{0}f(x) = f(x)$.

Definition 1.3 [12] Let $f \in L[a,b]$. The Left sided and right sided k-Riemann-Liouville integrals ${}_kJ_{a^+}^{\alpha}f$ and ${}_kJ_{b^-}^{\alpha}f$ of order $\frac{\alpha}{k}>0$ with $a \geqslant 0$ can be defined respectively by

$${}_kJ_{a^+}^{\alpha}f(x) = \frac{1}{\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1}f(t)dt, \qquad x>a$$

$${}_kJ_{b^-}^{\alpha}f(x) = \frac{1}{\Gamma_k(\alpha)}\int_x^b (t-x)^{\frac{\alpha}{k}-1}f(t)dt, \qquad x<b$$

here $\Gamma_k(\alpha)$ is the k-Gamma function defined as:

$$\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1}\exp^{-\frac{t^k}{k}}dt,$$

also

$$\Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha),$$

and ${}_1J^0_{a^+}f(x) = {}_1J^0_{b^-}f(x) = f(x)$
For $k = 1$, k-fractional integrals gives Riemann-Liouville integrals.

For some recent results connected with k-fractional integral inequalities (see, [1, 2, 8, 13, 18]).
In this paper, by means of the use of the Jensen-Mercer inequality, we prove Hermite-Hadamard's inequalities for k-fractional integrals and we mounted some new fractional inequalities related with the left part of Hermite-Hadamard type inequalities for differentiable mappings whose first derivatives in absolute value are convex.

2. Main results

By using the Jensen-Mercer inequality, Hermite-Hadamard's inequalities can be represented in k-fractional integral forms as follows.

Theorem 2.1 *Suppose $f : [a, b] \to R$ be a convex function, then*

$$\begin{aligned} f\Big(a+b-\frac{x+y}{2}\Big) &\leqslant \tfrac{\Gamma_k(\alpha+k)}{2(y-x)^{\frac{\alpha}{k}}}\Big[{}_kJ^{\alpha}_{(a+b-y)^+}f(a+b-x) + {}_kJ^{\alpha}_{(a+b-x)^-}f(a+b-y)\Big] \\ &\leqslant \tfrac{f(a+b-x)+f(a+b-y)}{2} \leqslant f(a)+f(b)-\tfrac{f(x)+f(y)}{2}, \end{aligned} \quad (2)$$

for all $x, y \in [a, b]$ and $\frac{\alpha}{k} > 0$.

Proof From the convexity of f we have

$$\begin{aligned} f\Big(a+b-\frac{\xi+\zeta}{2}\Big) &= f\Big(\frac{a+b-\xi+a+b-\zeta}{2}\Big) \\ &\leqslant \frac{1}{2}[f(a+b-\xi)+f(a+b-\zeta)], \end{aligned} \quad (3)$$

for all $\xi, \zeta \in [a, b]$. By changing of variables $a+b-\xi = t(a+b-x)+(1-t)(a+b-y)$ and $a+b-\zeta = (1-t)(a+b-x)+t(a+b-y)$ for $x, y \in [a, b]$ and $t \in [0, 1]$ in (3) we find that

$$\begin{aligned} &f\Big(a+b-\frac{x+y}{2}\Big) \\ &\quad\leqslant \frac{1}{2}[f(t(a+b-x)+(1-t)(a+b-y))+f((1-t)(a+b-x)+t(a+b-y))] \end{aligned} \quad (4)$$

Multiplying both sides of (4) by $t^{\frac{\alpha}{k}-1}$ and then integrating the resulting inequality with respect to t over $[0,1]$, we have

$$\begin{aligned}
&\frac{k}{\alpha} f\Big(a+b-\frac{x+y}{2}\Big)\\
&\leqslant \frac{1}{2}\Big[\int_o^1 t^{\frac{\alpha}{k}-1} f(t(a+b-x)+(1-t)(a+b-y))dt\\
&+\int_0^1 t^{\frac{\alpha}{k}-1} f((1-t)(a+b-x)+t(a+b-y))dt\Big]\\
&=\frac{1}{2(y-x)^{\frac{\alpha}{k}}}\Big[\int_{a+b-y}^{a+b-x}(u-(a+b-y))^{\frac{\alpha}{k}-1} f(u)du\\
&+\int_{a+b-y}^{a+b-x}((a+b-x)-u)^{\frac{\alpha}{k}-1} f(u)du\Big]\\
&=\frac{\Gamma_k(\alpha+k)}{2(y-x)^{\frac{\alpha}{k}}}\Big[{}_kJ^{\alpha}_{(a+b-y)^+} f(a+b-x)+{}_kJ^{\alpha}_{(a+b-x)^-} f(a+b-y)\Big].
\end{aligned}$$

And so

$$f\Big(a+b-\frac{x+y}{2}\Big)\leqslant \frac{\Gamma_k(\alpha)}{2(y-x)^{\frac{\alpha}{k}}}\Big[{}_kJ^{\alpha}_{(a+b-y)^+} f(a+b-x)+{}_kJ^{\alpha}_{(a+b-x)^-} f(a+b-y)\Big].$$

The proof of first inequality in (2) is completed. On the other hand, using the convexity of f we can write

$$f(t(a+b-x)+(1-t)(a+b-y))\leqslant tf(a+b-x)+(1-t)f(a+b-y)$$

$$f((1-t)(a+b-x)+t(a+b-y))\leqslant (1-t)f(a+b-x)+tf(a+b-y).$$

By adding these inequalities and using the Jensen-Mercer inequality, we have

$$\begin{aligned}
&f(t(a+b-x)+(1-t)(a+b-y))+f((1-t)(a+b-x)+t(a+b-y))\\
&\leqslant f(a+b-x)+f(a+b-y)\\
&\leqslant 2[f(a)+f(b)]-[f(x)+f(y)]. \qquad (5)
\end{aligned}$$

Multiplying both sides of (5) by $t^{\frac{\alpha}{k}-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain second and third inequalities in (2). ■

Remark 2.2 Under the conditions of Theorem 2.1 with $k=1$, then Theorem 2.1 reduces to inequalities (2.2) in [15, Theorem 2].

Remark 2.3 Under the assumptions of Theorem 2.1 with $\alpha=k=1$, then Theorem 2.1 reduces to [9, Theorem 2.1].

Remark 2.4 If in Theorem 2.1, we set $\alpha=k=1$, $x=a$ and $y=b$, then the inequalities (2) become the inequalities (1).

Remark 2.5 If we take $k=1$, $x=a$ and $y=b$ in Theorem 2.1, then Theorem 2.1 reduces in [16, Theorem 2].

Remark 2.6 Under the assumption of Theorem 2.1, if $x=a$ and $y=b$ then

$$f\Big(\frac{a+b}{2}\Big) \leqslant \frac{\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\Big[{}_kJ^{\alpha}_{a^+}f(b)+{}_kJ^{\alpha}_{b^-}f(a)\Big] \leqslant \frac{f(a)+f(b)}{2},$$

which is proved in [8].

Theorem 2.7 *Suppose* $f:[a,b]\to R$ *be a convex function, then*

$$\begin{aligned} & f\Big(a+b-\frac{x+y}{2}\Big) \\ & \leqslant \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(y-x)^{\frac{\alpha}{k}}}\Big[{}_kJ^{\alpha}_{(a+b-\frac{x+y}{2})^+}f(a+b-x)+{}_kJ^{\alpha}_{(a+b-\frac{x+y}{2})^-}f(a+b-y)\Big] \\ & \leqslant \frac{f(a+b-x)+f(a+b-y)}{2} \leqslant f(a)+f(b)-\frac{f(x)+f(y)}{2}, \end{aligned} \tag{6}$$

for all $x,y\in[a,b]$ *and* $\frac{\alpha}{k}>0$.

Proof To prove the first inequality of (6), by writing $\xi=\frac{t}{2}x+\frac{2-t}{2}y$ and $\zeta=\frac{2-t}{2}x+\frac{t}{2}y$ for $x,y\in[a,b]$ and $t\in[0,1]$ in inequality (3), we get

$$\begin{aligned} & 2f\Big(a+b-\frac{x+y}{2}\Big) \\ & \leqslant \frac{1}{2}\Big[f\Big(a+b-\Big(\frac{t}{2}x+\frac{2-t}{2}y\Big)\Big) \\ & +f\Big(a+b-\Big(\frac{2-t}{2}x+\frac{t}{2}y\Big)\Big)\Big]. \end{aligned} \tag{7}$$

And then multiplying both sides of (7) by $t^{\frac{\alpha}{k}-1}$ and then integrating the resulting inequality with respect to t over [0,1], we have

$$\begin{aligned} & \frac{2k}{\alpha}f\Big(a+b-\frac{x+y}{2}\Big) \\ & \leqslant \int_o^1 t^{\frac{\alpha}{k}-1}f\Big(a+b-\Big(\frac{t}{2}x+\frac{2-t}{2}y\Big)\Big)dt \\ & +\int_0^1 t^{\frac{\alpha}{k}-1}f\Big(a+b-\Big(\frac{2-t}{2}x+\frac{t}{2}y\Big)\Big)dt \\ & =\frac{2^{\frac{\alpha}{k}}}{(y-x)^{\frac{\alpha}{k}}}\Big[\int_{a+b-y}^{a+b-\frac{x+y}{2}}(u-(a+b-y))^{\frac{\alpha}{k}-1}f(u)du \\ & +\int_{a+b-\frac{x+y}{2}}^{a+b-x}((a+b-x)-u)^{\frac{\alpha}{k}-1}f(u)du\Big] \\ & =\frac{2^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{(y-x)^{\frac{\alpha}{k}}}\Big[{}_kJ^{\alpha}_{(a+b-\frac{x+y}{2})^-}f(a+b-y)+{}_kJ^{\alpha}_{(a+b-\frac{x+y}{2})^+}f(a+b-x)\Big]. \end{aligned}$$

And so

$$f\Big(a+b-\frac{x+y}{2}\Big) \leqslant \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha)}{(y-x)^{\frac{\alpha}{k}}}\Big[{}_kJ^{\alpha}_{(a+b-\frac{x+y}{2})^-}f(a+b-y)+{}_kJ^{\alpha}_{(a+b-\frac{x+y}{2})^+}f(a+b-x)\Big].$$

The first inequality of (6) is proved. For the proof of second inequality of (6), by using Jensen-Mercer inequality, we obtain

$$f\Big(a+b-\Big(\frac{t}{2}x+\frac{2-t}{2}y\Big)\Big) \leqslant f(a)+f(b)-\Big[\frac{t}{2}f(x)+\frac{2-t}{2}f(y)\Big]$$

$$f\Big(a+b-\Big(\frac{2-t}{2}x+\frac{t}{2}y\Big)\Big) \leqslant f(a)+f(b)-\Big[\frac{2-t}{2}f(x)+\frac{t}{2}f(y)\Big].$$

By adding these inequalities, we have

$$\begin{aligned} &f\Big(a+b-\Big(\frac{t}{2}x+\frac{2-t}{2}y\Big)\Big)+f\Big(a+b-\Big(\frac{2-t}{2}x+\frac{t}{2}y\Big)\Big) \\ &\leqslant 2[f(a)+f(b)]-\frac{f(x)+f(y)}{2}. \end{aligned} \tag{8}$$

Multiplying both sides of (8) by $t^{\frac{\alpha}{k}-1}$ and then integrating the resulting inequality with respect to t over $[a,b]$, we find second inequality of (6). ■

Remark 2.8 If we take $k=1$ in Theorem 2.7, then Theorem 2.7 reduces in [15, Theorem 3].

Remark 2.9 Under the assumptions of Theorem 2.7 with $\alpha=k=1$, then Theorem 2.7 reduces to [9, Theorem 2.1].

Remark 2.10 If in Theorem 2.7, we set $\alpha=k=1$, $x=a$ and $y=b$, then the inequalities (6) reduces to the inequalities (1).

Remark 2.11 If we set $k=1$, $x=a$ and $y=b$ in Theorem 2.7, then Theorem 2.7 reuces to [17, Theorem 3].

Corollary 2.1 If we choose $x=a$ and $y=b$ in Theorem 2.7, then we have following inequalities for k-Riemann-Liouville fractional integrals:

$$\begin{aligned} f\Big(\frac{a+b}{2}\Big) &\leqslant \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\Big[{}_kJ^{\alpha}_{(\frac{a+b}{2})^+}f(b)+{}_kJ^{\alpha}_{(\frac{a+b}{2})^-}f(a)\Big] \\ &\leqslant \frac{f(a)+f(b)}{2}. \end{aligned}$$

.

Lemma 2.12 *Let $f:[a,b]\to R$ be a differentiable mapping on (a,b) with $a<b$. If $f^{'}\in L[a,b]$, then the following identity for $k-$fractional integrals holds:*

$$\begin{aligned} &\frac{f(a+b-x)+f(a+b-y)}{2}-\frac{\Gamma_k(\alpha+k)}{2(y-x)^{\frac{\alpha}{k}}}\Big[{}_kJ^{\alpha}_{(a+b-y)^+}f(a+b-x)+{}_kJ^{\alpha}_{(a+b-x)^-}f(a+b-y)\Big] \\ &=\frac{y-x}{2}\int_0^1(t^{\frac{\alpha}{k}}-(1-t)^{\frac{\alpha}{k}})f^{'}(a+b-(tx+(1-t)y))dt, \end{aligned}$$

for all $x,y\in[a,b]$, $\frac{\alpha}{k}>0$ and $t\in[0,1]$.

Proof It suffices to note that

$$\begin{aligned}I &= \int_0^1 (t^{\frac{\alpha}{k}} - (1-t)^{\frac{\alpha}{k}})f'(a+b-(tx+(1-t)y))dt\\ &= \int_0^1 t^{\frac{\alpha}{k}} f'(a+b-(tx+(1-t)y))dt - \int_0^1 (1-t)^{\frac{\alpha}{k}} f'(a+b-(tx+(1-t)y))dt\\ &= I_1 - I_2.\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}I_1 &= \int_0^1 t^{\frac{\alpha}{k}} f'(a+b-(tx+(1-t)y))dt\\ &= \frac{t^{\frac{\alpha}{k}} f(a+b-(tx+(1-t)y)}{y-x}\Big|_0^1 - \frac{\alpha}{k(y-x)}\int_0^1 t^{\frac{\alpha-k}{k}} f(a+b-(tx+(1-t)y))dt\\ &= \frac{f(a+b-x)}{y-x} - \frac{\Gamma_k(\alpha+k)}{(y-x)^{\frac{\alpha+k}{k}}}\, {}_kJ^{\alpha}_{(a+b-x)^-} f(a+b-y).\end{aligned}$$

Similarly, we get

$$\begin{aligned}I_2 &= \int_0^1 (1-t)^{\frac{\alpha}{k}} f'(a+b-(tx+(1-t)y))dt\\ &= \frac{(1-t)^{\frac{\alpha}{k}} f(a+b-(tx+(1-t)y)}{y-x}\Big|_0^1 + \frac{\alpha}{k(y-x)}\int_0^1 t^{\frac{\alpha-k}{k}} f(a+b-(tx+(1-t)y))dt\\ &= -\frac{f(a+b-x)}{y-x} + \frac{\Gamma_k(\alpha+k)}{(y-x)^{\frac{\alpha+k}{k}}}\, {}_kJ^{\alpha}_{(a+b-y)^+} f(a+b-x).\end{aligned}$$

We can write

$$\begin{aligned}I &= I_1 - I_2\\ &= \frac{f(a+b-x)+f(a+b-y)}{y-x} - \frac{\Gamma_k(\alpha+k)}{(y-x)^{\frac{\alpha+k}{k}}}\\ &\times \left[{}_kJ^{\alpha}_{(a+b-y)^+} f(a+b-x) + {}_kJ^{\alpha}_{(a+b-x)^+} f(a+b-y)\right].\end{aligned}$$

Multiplying both sides by $\frac{y-x}{2}$, we have the desired identity. ■

Remark 2.13 Under assumption of Lemma 2.12, if $\alpha = k = 1$, $x = a$ and $y = b$ then

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)du = \frac{b-a}{2}\int_0^1 (2t-1)f'((1-t)a+tb)dt,$$

which is proved in [5].

Remark 2.14 Under the assumption of Lemma 2.12, if $x = a$ and $y = b$ then

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\Big[{}_kJ^{\alpha}_{a^+}f(b) + {}_kJ^{\alpha}_{b^-}f(a)\Big]$$
$$= \frac{b-a}{2}\int_0^1 (t^{\frac{\alpha}{k}} - (1-t)^{\frac{\alpha}{k}})f'((1-t)a+tb)dt,$$

which is proved in [8].

Remark 2.15 Under assumption of Lemma 2.12, if $k = 1$, $x = a$ and $y = b$ then

$$\frac{f(a)+(b}{2} - \frac{\Gamma(\alpha+1)}{2(y-x)}\Big[J^{\alpha}_{a^+}f(b) + J^{\alpha}_{b^-}f(a)\Big] = \frac{b-a}{2}\int_0^1 (t^{\alpha} - (1-t)^{\alpha})f'((1-t)a+tb))dt,$$

which is proved in [16].

Remark 2.16 If we take $\alpha = k = 1$, in Lemma 2.12, then Lemma 2.12 gives [15, Corollory 1].

Theorem 2.17 *Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality for $k-$fractional integrals holds:*

$$\Bigg|\frac{f(a+b-x)+f(a+b-y)}{2} - \frac{\Gamma_k(\alpha+k)}{2(y-x)^{\frac{\alpha}{k}}}\Big[{}_kJ^{\alpha}_{(a+b-y)^+}f(a+b-x) + {}_kJ^{\alpha}_{(a+b-x)^+}f(a+b-y)\Big]\Bigg|$$
$$\leqslant \frac{y-x}{\alpha+k}\Big(k - \frac{k}{2^{\frac{\alpha}{k}}}\Big)\Big[|f'(a)| + |f'(b)| - \frac{|f'(x)|+|f'(y)|}{2}\Big]$$

for all $x, y \in [a,b]$ and $\frac{\alpha}{k} > 0$.

Proof By means of Lemma 2.12 and jensen-mercer inequality, we find that

$$\Bigg|\frac{f(a+b-x)+f(a+b-y)}{2}$$
$$- \frac{\Gamma_k(\alpha+k)}{2(y-x)^{\frac{\alpha}{k}}}\Big[{}_kJ^{\alpha}_{(a+b-y)^+}f(a+b-x) + {}_kJ^{\alpha}_{(a+b-x)^+}f(a+b-y)\Big]\Bigg|$$
$$\leqslant \frac{y-x}{2}\int_0^1 |t^{\frac{\alpha}{k}} - (1-t)^{\frac{\alpha}{k}}||f'(a+b-(tx+(1-t)y))|dt$$
$$\leqslant \frac{y-x}{2}\int_0^1 |t^{\frac{\alpha}{k}} - (1-t)^{\frac{\alpha}{k}}|[|f'(a)| + |f'(b)| - (t|f'(x)| + (1-t)|f'(y)|)]dt$$
$$= \frac{y-x}{2}\Bigg\{\int_0^{\frac{1}{2}} ((1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}}|)[|f'(a)| + |f'(b)| - (t|f'(x)| + (1-t)|f'(y)|)]dt$$
$$+ \int_{\frac{1}{2}}^1 (t^{\frac{\alpha}{k}} - (1-t)^{\frac{\alpha}{k}}|)[|f'(a)| + |f'(b)| - (t|f'(x)| + (1-t)|f'(y)|)]dt\Bigg\}$$
$$= \frac{y-x}{2}(S_1 + S_2)$$

Calculating S_1 and S_2, we obtain

$$S_1 = (|f^{'}(a)| + |f^{'}(b)|) \int_0^{\frac{1}{2}} ((1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}} dt - \Big\{ |f^{'}(x)| \Big[\int_0^{\frac{1}{2}} t(1-t)^{\frac{\alpha}{k}} dt - \int_0^{\frac{1}{2}} t^{\frac{\alpha+k}{k}} dt \Big]$$
$$+ |f^{'}(y)| \Big[\int_0^{\frac{1}{2}} (1-t)^{\frac{\alpha+k}{k}} dt - \int_0^{\frac{1}{2}} (1-t) t^{\frac{\alpha+k}{k}} dt \Big] \Big\}$$
$$= (|f^{'}(a)| + |f^{'}(b)|) \Big(\frac{k}{\alpha+k} - \frac{(\frac{k}{2^{\frac{\alpha}{k}}})}{\alpha+k} \Big) - \Big\{ |f^{'}(x)| \Big(\frac{k^2}{(\alpha+k)(\alpha+2k)} - \frac{(\frac{k}{2^{\frac{\alpha+k}{k}}})}{\alpha+k} \Big)$$
$$+ |f^{'}(y)| \Big(\frac{k}{\alpha+2k} - \frac{(\frac{k}{2^{\frac{\alpha+k}{k}}})}{\alpha+k} \Big) \Big\}$$

and

$$S_2 = (|f^{'}(a)| + |f^{'}(b)|) \int_{\frac{1}{2}}^{1} (t^{\frac{\alpha}{k}} - (1-t)^{\frac{\alpha}{k}} dt - \Big\{ |f^{'}(x)| \Big[\int_{\frac{1}{2}}^{1} t^{\frac{\alpha+k}{k}} dt - \int_{\frac{1}{2}}^{1} t(1-t)^{\frac{\alpha}{k}} dt \Big]$$
$$+ |f^{'}(y)| \Big[\int_{\frac{1}{2}}^{1} (1-t)^{\frac{\alpha+k}{k}} dt - \int_{\frac{1}{2}}^{1} (1-t) t^{\frac{\alpha+k}{k}} dt \Big] \Big\}$$
$$= (|f^{'}(a)| + |f^{'}(b)|) \Big(\frac{k}{\alpha+k} - \frac{(\frac{k}{2^{\frac{\alpha}{k}}})}{\alpha+k} \Big) - \Big\{ |f^{'}(x)| \Big(\frac{k}{\alpha+2k} - \frac{(\frac{k}{2^{\frac{\alpha+k}{k}}})}{\alpha+k} \Big)$$
$$+ |f^{'}(y)| \Big(\frac{k^2}{(\alpha+k)(\alpha+2k)} - \frac{(\frac{k}{2^{\frac{\alpha+k}{k}}})}{\alpha+k} \Big) \Big\}.$$

By adding S_1 and S_2, we have the conclusion Theorem 2.17. ■

Remark 2.18 Under the assumption of Theorem 2.17, if $\alpha = k = 1$, $x = a$ and $y = b$ then

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(u)du \right| \leqslant \frac{b-a}{8} \Big(|f^{'}(a)| + |f^{'}(b)| \Big),$$

which is proved in [5].

Remark 2.19 If we take $k = 1$ in Theorem 2.17, then Theorem 2.17 reduces in [15, Theorm 4] .

Remark 2.20 Under the assumption of Theorem 2.17, if $x = a$ and $y = b$ then

$$\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}} \Big[{}_kJ^{\alpha}_{(a)^+} f(b) + {}_kJ^{\alpha}_{(b)^-} f(a) \Big] \right|$$
$$\leqslant \frac{b-a}{\alpha+k} \Big(k - \frac{k}{2^{\frac{\alpha}{k}}} \Big) \Big(\frac{|f^{'}(a)| + |f^{'}(b)|}{2} \Big),$$

which is proved in [8].

Remark 2.21 Under the assumption of Theorem 2.17, if $k = 1$, $x = a$ and $y = b$

then

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{(a)^+}f(b)+J^{\alpha}_{(b)^-}f(a)\right]\right|$$
$$\leqslant \frac{b-a}{2(\alpha+1)}\Big(1-\frac{1}{2^{\alpha}}\Big)\Big(|f^{'}(a)|+|f^{'}(b)|\Big),$$

which is proved in [16].

Lemma 2.22 *Let $f:[a,b]\to R$ be a differentiable mapping on (a,b) with $a<b$. if $f^{'}\in L[a,b]$, then the following equality for $k-$fractional integrals holds:*

$$\frac{2^{\frac{\alpha-k}{k}}\Gamma_k(\alpha+k)}{2(y-x)^{\frac{\alpha}{k}}}\Big[{}_kJ^{\alpha}_{(a+b-\frac{x+y}{2})^+}f(a+b-x)+{}_kJ^{\alpha}_{(a+b-\frac{x+y}{2})^-}f(a+b-y)\Big]-f\Big(a+b-\frac{x+y}{2}\Big)$$
$$=\frac{y-x}{4}\int_0^1 t^{\frac{\alpha}{k}}\Big[f^{'}\Big(a+b-\Big(\frac{2-t}{2}x+\frac{t}{2}y\Big)\Big)-f^{'}\Big(a+b-\Big(\frac{t}{2}x+\frac{2-t}{2}y\Big)\Big)\Big]dt,$$

for all $x,y\in[a,b]$, $\frac{\alpha}{k}>0$ and $t\in[0,1]$.

Proof It can be prove similar to the proof of Lemma 2.12. ∎

Remark 2.23 If we take $k=1$ in Lemma 2.22, then Lemma 2.22 reduces in [15, Lemma 2].

Remark 2.24 If we take $k=1$ along with $x=a$ and $y=b$ in Lemma 2.22, then Lemma 2.22 reduces in [17, Lemma 3].

Theorem 2.25 *Let $f:[a,b]\to R$ be a differentiable mapping on (a,b) with $a<b$. If $f^{'}\in L[a,b]$, then the following equality for $k-$fractional integrals holds:*

$$\left|\frac{2^{\frac{\alpha-k}{k}}\Gamma_k(\alpha+k)}{2(y-x)^{\frac{\alpha}{k}}}\Big[{}_kJ^{\alpha}_{(a+b-\frac{x+y}{2})^+}f(a+b-x)+{}_kJ^{\alpha}_{(a+b-\frac{x+y}{2})^-}f(a+b-y)\Big]-f\Big(a+b-\frac{x+y}{2}\Big)\right|$$
$$\leqslant \frac{k(y-x)}{2(\alpha+k)}\Big[|f^{'}(a)|+|f^{'}(x)|-\frac{|f^{'}(x)|+|f^{'}(y)|}{2}\Big],$$

for all x,y $\in[a,b]$ and $\frac{\alpha}{k}>0$.

Proof Using the Lemma 2.22 and Jensen-mercer inequality, we find

$$\left|\frac{2^{\frac{\alpha-k}{k}}\Gamma_k(\alpha+k)}{2(y-x)^{\frac{\alpha}{k}}}\Big[{}_kJ^{\alpha}_{(a+b-\frac{x+y}{2})^+}f(a+b-x)+{}_kJ^{\alpha}_{(a+b-\frac{x+y}{2})^-}f(a+b-y)\Big]-f\Big(a+b-\frac{x+y}{2}\Big)\right|$$
$$\leqslant \frac{y-x}{4}\Big\{\int_0^1 t^{\frac{\alpha}{k}}\Big|f^{'}\Big(a+b-\Big(\frac{2-t}{2}x+\frac{t}{2}y\Big)\Big)\Big|dt+\int_0^1 t^{\frac{\alpha}{k}}\Big|f^{'}\Big(a+b-\Big(\frac{t}{2}x+\frac{2-t}{2}y\Big)\Big)\Big|dt\Big\}$$
$$\leqslant \frac{y-x}{4}\Big\{\int_0^1 t^{\frac{\alpha}{k}}\Big[|f^{'}(a)|+|f^{'}(b)|-\Big(\frac{2-t}{2}|f^{'}(x)|+\frac{t}{2}|f^{'}(y)|\Big)\Big]dt$$
$$+\int_0^1 t^{\frac{\alpha}{k}}\Big[|f^{'}(a)|+|f^{'}(b)|-\Big(\frac{t}{2}|f^{'}(x)|+\frac{2-t}{2}|f^{'}(y)|\Big)\Big]dt\Big\}$$
$$=\frac{k(y-x)}{2(\alpha+k)}\Big[|f^{'}(a)|+|f^{'}(b)|-\frac{|f^{'}(x)|+|f^{'}(y)|}{2}\Big],$$

which completed the proof. ∎

Remark 2.26 If we take $k=1$ in Theorem 2.25, then Theorem 2.25 reduces in [15, Theorm 5].

Remark 2.27 If we take $\alpha = 1$ along with $k = 1$ in Theorem 2.25, then Theorem 2.25 gives [15, Corollary 2].

Theorem 2.28 *Let $f : [a, b] \to R$ be a differentiable mapping on (a, b) with $a < b$. if $|f'|^q$ is convex on $[a, b]$, $q > 1$, then for all $x, y \in [a, b]$ and $\frac{\alpha}{k} > 0$, the following inequality for $k-$fractional integrals holds:*

$$\left|\frac{2^{\frac{\alpha-k}{k}}\Gamma_k(\alpha+k)}{2(y-x)^{\frac{\alpha}{k}}}\Big[{}_kJ^{\alpha}_{(a+b-\frac{x+y}{2})^+}f(a+b-x)+{}_kJ^{\alpha}_{(a+b-\frac{x+y}{2})^-}f(a+b-y)\Big]-f\Big(a+b-\frac{x+y}{2}\Big)\right|$$
$$\leqslant \frac{(y-x)}{4}\Big(\frac{k}{\alpha p+k}\Big)^{\frac{1}{p}}\Big[\Big(|f'(a)|^q+|f'(b)|^q-\frac{3|f'(x)|^q+|f'(y)|^q}{4}\Big)^{\frac{1}{q}}$$
$$+\Big(|f'(a)|^q+|f'(b)|^q-\frac{|f'(x)|^q+3|f'(y)|^q}{4}\Big)^{\frac{1}{q}}\Big],$$

where $\frac{1}{q}+\frac{1}{p}=1$.

Proof From Lemma 2.22, using the Hölder's inequality, we have

$$\left|\frac{2^{\frac{\alpha-k}{k}}\Gamma_k(\alpha+k)}{2(y-x)^{\frac{\alpha}{k}}}\Big[{}_kJ^{\alpha}_{(a+b-\frac{x+y}{2})^+}f(a+b-x)+{}_kJ^{\alpha}_{(a+b-\frac{x+y}{2})^-}f(a+b-y)\Big]-f\Big(a+b-\frac{x+y}{2}\Big)\right|$$
$$\leqslant \frac{y-x}{4}\Big(\int_0^1 t^{\frac{\alpha p}{k}}dt\Big)^{\frac{1}{p}}\Big\{\Big(\int_0^1\Big|f'\Big(a+b-\Big(\frac{2-t}{2}x+\frac{t}{2}y\Big)\Big)\Big|^q dt\Big)^{\frac{1}{q}}$$
$$+\Big(\int_0^1\Big|f'\Big(a+b-\Big(\frac{t}{2}x+\frac{2-t}{2}y\Big)\Big)\Big|^q dt\Big)^{\frac{1}{q}}\Big\}.$$

Using Jensen-Mercer inequality because of the convexity of $|f'|$, we have

$$\left|\frac{2^{\frac{\alpha-k}{k}}\Gamma_k(\alpha+k)}{2(y-x)^{\frac{\alpha}{k}}}\Big[{}_kJ^{\alpha}_{(a+b-\frac{x+y}{2})^+}f(a+b-x)+{}_kJ^{\alpha}_{(a+b-\frac{x+y}{2})^-}f(a+b-y)\Big]-f\Big(a+b-\frac{x+y}{2}\Big)\right|$$
$$\leqslant \frac{y-x}{4}\Big(\frac{k}{\alpha p+k}\Big)^{\frac{1}{p}}\Big\{\Big(\int_0^1|f'(a)|^q+|f'(b)|^q-\Big(\frac{2-t}{2}|f'(x)|^q+\frac{t}{2}|f'(y)|^q\Big)\Big)dt\Big)^{\frac{1}{q}}$$
$$+\Big(\int_0^1|f'(a)|^q+|f'(b)|^q-\Big(\frac{2-t}{2}|f'(x)|^q+\frac{t}{2}|f'(y)|^q\Big)\Big)dt\Big)^{\frac{1}{q}}\Big\}$$
$$=\frac{y-x}{4}\Big(\frac{k}{\alpha p+k}\Big)^{\frac{1}{p}}$$
$$\times\Big[\Big(|f'(a)|^q+|f'(b)|^q-\frac{3|f'(x)|^q+|f'(y)|^q}{4}\Big)^{\frac{1}{q}}+\Big(|f'(a)|^q+|f'(b)|^q-\frac{|f'(x)|^q+3|f'(y)|^q}{4}\Big)^{\frac{1}{q}}\Big],$$

and so the proof is completed. ■

Remark 2.29 If we take $k = 1$, $x = a$ and $y = b$ in Theorem 2.28, then Theorem 2.28 reduces in [17, Theorm 6].

Remark 2.30 If we take $k = 1$ in Theorem 2.28, then Theorem 2.28 reduces in [15, Theorm 6].

3. Conclusions

In this paper, we prove some new inequalities of Hermite–Hadamard-Mercer like associated with k-fractional integrals. The results of this paper generalize the several obtained results in the field of Hermite–Hadamard and Hermite–Hadamard-Mercer type inequalities. We hope that the ideas and techniques of this paper will inspire to interested readers working in this direction.

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